Metric Distortion Under Probabilistic Voting

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Abstract

Metric distortion in social choice provides a framework for assessing how well voting rules minimize social cost in scenarios where voters and candidates exist in a shared metric space, with voters submitting rankings and the rule outputting a single winner. We expand this framework to include probabilistic voting. Our extension encompasses a broad range of probability functions, including widely studied models like Plackett-Luce (PL) and Bradley-Terry, and a novel "pairwise quantal voting" model inspired by quantal response theory. We demonstrate that distortion results under probabilistic voting better correspond with conventional intuitions regarding popular voting rules such as Plurality, Copeland, and Random Dictator (RD) than those under deterministic voting. For example, in the PL model with candidate strength inversely proportional to the square of their metric distance, we show that Copeland's distortion is at most 2, whereas that of RD is $\Omega(\sqrt{m})$ in large elections, where m is the number of candidates. This contrasts sharply with the classical model, where RD beats Copeland with a distortion of 3 versus 5 [1].

1 Introduction

Societies must make decisions collectively; different agents often have conflicting interests, and the choice of the mechanism used for combining everyone's opinions often makes a big difference to the outcome. The machine learning community has applied social choice principles for AI alignment [2, 3], algorithmic fairness [4, 5], and preference modelling [6, 7]. Over the last century, there has been increasing interest in using computational tools to analyse and design voting rules [8–11]. One prominent framework for evaluating voting rules is that of *distortion* [12], where the voting rule has access to only the *ordinal* preferences of the voters. However, the figure of merit is the sum of all voters' *cardinal* utilities (or costs). The distortion of a voting rule is the worst-case ratio of the cost of the alternative it selects and the cost of the truly optimal alternative.

An additional assumption is imposed in *metric distortion* [1] – that the voters and candidates all lie in a shared (unknown) metric space, and costs are given by distances (thus satisfying non-negativity and triangular inequality). This model is a generalization of a commonly studied *spatial model of voting* in the Economics literature [13, 14], and has a natural interpretation of voters liking candidates with a similar ideological position across many dimensions. While metric distortion is a powerful framework and has led to the discovery and re-discovery of interesting voting rules (e.g. Plurality Veto [15] and the study of Maximal Lotteries [16] for metric distortion by Charikar et al. [17]), its outcomes sometimes do not correspond with traditional wisdom around popular voting rules. For example, the overly simple *Random Dictator (RD)* rule (where the winner is the top choice of a uniform randomly selected voter) beats the *Copeland* rule (which satisfies the Condorcet Criterion [10] and other desirable properties) with a metric distortion of 3 versus 5 [1].

While not yet adopted in the metric distortion framework, there is a mature line of work on *Probabilistic voting* (PV) [18–20]. Here, the focus is on the behavioural modelling of voters and

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accounting for the randomness of their votes. Two sources of this randomness often cited in the literature are the boundedness of the voters' rationality and the noise in their estimates of candidates' positions. A popular model for this behaviour is based on the *Quantal Response Theory* [20]. Another closely related line of work is on Random Utility Models (RUMs) [21–23] in social choice where the hypothesis is that the candidates have ground-truth strengths. Voters make noisy observations of these strengths and vote accordingly. We adopt these models of voting behaviour and study it within the metric distortion framework. The questions we ask are:

Given a model of probabilistic voting, what is the metric distortion of popular voting rules?
How does this differ (qualitatively and quantitatively) from the deterministic model?

1.1 Preliminaries and Notation

Let $\mathcal{N}$ be a set of n voters and $\mathcal{A}$ be the set of m candidates. Let $\mathcal{S}$ be the set of total orders on $\mathcal{A}$. Each voter $i \in \mathcal{N}$ has a preference ranking $\sigma_i \in \mathcal{S}$. A vote profile is a set of preference rankings $\sigma_{\mathcal{N}} = (\sigma_1, ..., \sigma_n) \in \mathcal{S}^n$ for all voters. The tuple $(\mathcal{N}, \mathcal{A}, \sigma_{\mathcal{N}})$ defines an instance of an election. Let $\Delta(\mathcal{A})$ denote the set of all probability distributions over the set of candidates.

Definition 1 (Voting Rule)**.** *A voting rule* $f : S^n \to \Delta(\mathcal{A})$ *takes a vote profile* σ_N *and outputs a probability distribution* p *over the alternatives.*

For deterministic voting rules, we overload notation by saying that the rule's output is a candidate and not a distribution. We now define some voting rules [10]. Let $\mathbb{I}$ denote the indicator function.

Random Dictator Rule: Select a voter uniformly at random and output their top choice, i.e., $\text{RD}(\sigma_{\mathcal{N}}) = p$ such that $p_j = \frac{1}{n} \sum_{i \in \mathcal{N}} \mathbb{I}(\sigma_{i,1} = j)$.

Plurality Rule: Choose the candidate who is the top choice of the most voters, i.e., $\text{PLU}(\sigma_{\mathcal{N}}) = \arg\max_{j \in \mathcal{A}} \sum_{i \in \mathcal{N}} \mathbb{I}(\sigma_{i,1} = j)$. Ties are broken arbitrarily.

Copeland Rule: Choose the candidate who wins the most pairwise comparisons, i.e., $\text{COP}(\sigma_{\mathcal{N}}) = \arg\max_{j \in \mathcal{A}} \sum_{j' \in \mathcal{A} \setminus \{j\}} \mathbb{I}\left(\sum_{i \in \mathcal{N}} \mathbb{I}(j \succ_{\sigma_i} j') > \frac{n}{2}\right)$. Ties are broken arbitrarily.

Distance function $d : (\mathcal{N} \cup \mathcal{A})^2 \to \mathbb{R}_{\geq 0}$ satisfies triangular inequality ($d(x, y) \leq d(x, z) + d(z, y)$) and symmetry ($d(x, y) = d(y, x)$). The distance between voter $i \in \mathcal{N}$ and candidate $j \in \mathcal{A}$ is also referred to as the *cost* of j for i. We consider the most commonly studied social cost function, which is the sum of the costs of all voters. $SC(j, d) := \sum_{i \in \mathcal{N}} d(i, j)$.

In deterministic voting, the preference ranking σ_i of voter i is consistent with the distances. That is, $d(i, j) > d(i, j') \implies j' \succ_{\sigma_i} j$ for all voters $i \in \mathcal{N}$ and candidates $j, j' \in \mathcal{A}$. Let $\rho(\sigma_{\mathcal{N}})$ be the set of distance functions d consistent with vote profile $\sigma_{\mathcal{N}}$. The metric distortion of a voting rule is:

Definition 2 (Metric Distortion)**.** $\text{DIST}(f) = \sup_{\mathcal{N}, \mathcal{A}, \sigma_{\mathcal{N}}} \sup_{d \in \rho(\sigma_{\mathcal{N}})} \frac{\mathbb{E}[SC(f(\sigma_{\mathcal{N}}), d)]}{\min_{j \in \mathcal{A}} SC(j, d)}$.

1.2 Our Contributions

We extend the study of metric distortion to probabilistic voting (Definition 4). This extension is useful since voters, in practice, have been shown to vote randomly [20]. We define axiomatic properties of models of probabilistic voting which are suitable for studying metric distortion. These are scale-freeness with distances (Axiom 1), pairwise order probabilities being independent of other candidates (Axiom 2), and strict monotonicity of pairwise order probabilities in distances (Axiom 3).

All our results apply to a broad class of models of probabilistic models, as explained in § 2. We provide distortion bounds for all $n \geq 3$ and $m \geq 2$, which are most salient in the limit $n \to \infty$. For large elections (m fixed, $n \to \infty$), we provide matching upper and lower bounds on the distortion of Plurality, an upper bound for Copeland, and a lower bound for RD. The distortion of plurality grows linearly in m. The distortion upper bound of Copeland is constant. The distortion lower bound for RD increases sublinearly in m where this rate depends on the probabilistic model. Crucially, our results match those in deterministic voting in the limit where the randomness goes to zero.

The technique is as follows. For the problem of maximizing the distortion, we establish a critical threshold of the expected fraction of votes on pairwise comparisons on all edges on a directed path from a winner to the "true optimal" candidate for Copeland and Plurality. This path is one or two hops for Copeland and one for Plurality. We then formulate a linear-fractional program which incorporates this critical threshold. We linearize this program via the sub-level sets technique [24], and find a feasible solution of the dual problem. Concentration inequalities on this solution provide an upper bound on the distortion. We find a matching lower bound for Plurality by construction.

1.3 Related Work

Metric distortion Anshelevich et al. [1] initiated the study of metric distortion and showed that any deterministic voting rule has a distortion of at least 3 and that Copeland has a distortion of 5. The Plurality Veto Rule attains the optimal distortion of 3 [15]. Charikar and Ramakrishnan [25] showed that any randomized voting rule has a distortion of at least 2.112. Charikar et al. [17] gave a randomized voting rule with a distortion of at most 2.753. Anshelevich et al. [26] gave a useful survey on distortion in social choice.

Distortion with Additional Information Abramowitz et al. [27] showed that deterministic voting rules achieve a distortion of 2 when voters provide preference strengths as ratios of distances. Amanatidis et al. [28] demonstrated that even a few queries from each voter can significantly improve distortion in non-metric settings. Anshelevich et al. [29] examined threshold approval voting, where voters approve candidates with utilities above a threshold. Our work relates to these studies since in probabilistic voting, the likelihood of a voter switching the order of two candidates depends on the relative strength of their preference, often resulting in lower distortion than in deterministic voting.

Probabilistic voting and random utility models (RUMs) Hinich [30] showed that the celebrated Median Voter Theorem of [31] does not hold under probabilistic voting. Classical work has focused on studying the equilibrium positions of voters and/or candidates in game-theoretic models of probabilistic voting [20, 32–35]. McKelvey and Patty [20] adopt the quantal response model, a popular way to model agents' bounded rationality.

RUMs have mostly been studied in social choice [21, 23, 36] with the hypothesis that candidates have *universal* ground-truth strengths, which voters make noisy observations of. Our model is the same as RUM regarding the voters' behaviour; however, voters have *independent* costs from candidates. The Plackett-Luce (PL) model [37, 38] has been widely studied in social choice [39–41]. For probabilities on pairwise orders, PL reduces to the Bradley-Terry (BT) model [42]. These probabilities are proportional to candidates' strengths (which we define as the inverse of powers of costs).

The widely studied Mallows model [43], based on Condorcet [44], flips the order of each candidate pair (relative to a ground truth ranking) with a constant probability $p \in (0, \frac{1}{2})$ [45, 46]. The process is repeated if a linear order is not attained. In the context of metric distortion, a limitation of this model is that it doesn't account for the relative distance of candidates to the voter. For a comprehensive review of RUM models, see Marden [47]. Critchlow et al. [48] does an axiomatic study of RUM models; our axioms are grounded in metric distortion and are distinct from theirs.

Recently, there has been significant interest in smoothed analysis [49] of social choice. Here, a small amount of randomness is added to problem instances and its effect is studied on the satisfiability of axioms [50–53] and the computational complexity of voting rules [54–56]. Baumeister et al. [50] term this model as being 'towards reality,' highlighting the need to study the randomness in the election instance generation processes. Unlike smoothed analysis where the voter and candidate positions are randomized, we consider these positions fixed, but the submitted votes are random given these positions. The technical difference appears in the benchmark (the "optimal" outcome in the denominator of the distortion is unchanged in our framework and changes in smoothed analysis).

2 Axioms and Model

Under probabilistic voting, the submitted preferences may no longer be consistent with the underlying distances. For a distribution $\mathcal{P}(d)$ over $\sigma_{\mathcal{N}}$, let $q^{\mathcal{P}(d)}(i, j, j')$ denote the induced marginal probability that voter i ranks candidate j higher than j'. We focus on these marginal probabilities on pairwise orders and provide axioms for classifying which $q^{\mathcal{P}(d)}(\cdot)$ are suitable for studying distortion.

Axiom 1 (Scale-Freeness (SF))**.** *The probability $q^{\mathcal{P}(d)}(\cdot)$ must be invariant to scaling of d. That is, for any tuple (i, j, j') and any constant $\kappa > 0$, we must have $q^{\mathcal{P}(d)}(i, j, j') = q^{\mathcal{P}(\kappa d)}(i, j, j')$.*

Table 1: Axioms satisfied by commonly studied models of probabilistic voting

	Axiom 1: SF	Axiom 2: IOC	Axiom 3: Strict Monotonocity
Mallows	✓	×	×
PL/BT with exponential in d	×	✓	✓
PL/BT with powers of d	✓	✓	✓
PQV	✓	✓	✓

Note that the metric distortion (Definition 2) for deterministic voting is scale-free. We want to retain the same property in the probabilistic model as well. Conceptually, one may think of the voter's preferences as being a function of the relative (and not absolute) distances to the candidates.

Axiom 2 (Independence of Other Candidates (IOC))**.** *The probability $q^{\mathcal{P}(d)}(i,j,j')$ must be independent of the distance of voter i to all 'other' candidates, i.e., those in $\mathcal{A}\setminus\{j,j'\}$.*

This axiom extends Luce's choice axioms [38], defined for selecting the top choice, to entire rankings. IOC is reminiscent of the *independence of irrelevant alternatives* axiom for voting rules.

Axiom 3 (Strict Monotonicity (SM))**.** *For every tuple (i,j,j'), for fixed distance $d(i,j)>0$, the probability $q^{\mathcal{P}(d)}(i,j,j')$ must be strictly increasing in $d(i,j')$ at all but at most finitely many points.*

The monotonicity in $d(i,j)$ follows since $q^{\mathcal{P}(d)}(i,j',j)=1-q^{\mathcal{P}(d)}(i,j,j')$. This axiom is natural.

In the Mallows model [43], $q^{\mathcal{P}(d)}(\cdot)$ was derived by Busa-Fekete et al. [57] and is as follows:

$$\text{Mallows:}\qquad q^{\mathcal{P}(d)}(i,j,j')=h(r_{j'}-r_j+1,\phi)-h(r_{j'}-r_j,\phi). \tag{1}$$

Here $h(k,\phi)=\frac{k}{(1-\phi^k)}$. Whereas r_j and $r_{j'}$ are the positions of j and j' in the ground-truth (noiseless) ranking, and the constant ϕ is a dispersion parameter. Observe that this model fails Axiom 2 since it depends on the number of candidates between j and j' in the noiseless ranking. It also fails Axiom 3 since it does not depend on the exact distances but only on the order of the distances.

Plackett-Luce Model: The PL model [37, 38] is 'sequential' in the following way. For each voter $i\in\mathcal{N}$, each candidate $j\in\mathcal{A}$ has a 'strength' $s_{i,j}$. In most of the literature on RUMs, a common assumption is that $s_{i,j}$ is the same for all voters i. However, we choose this more general model to make it useful in the context of metric distortion. The voter chooses their top choice with probability proportional to the strengths. Similarly, for every subsequent rank, they choose a candidate from among the *remaining* ones with probabilities proportional to their strengths. In terms of the pairwise order probabilities, the PL model reduces to the Bradley-Terry (BT) model [42], that is:

$$\text{PL/BT:}\qquad q^{\mathcal{P}(d)}(i,j,j')=\frac{s_{i,j}}{s_{i,j}+s_{i,j'}}. \tag{2}$$

Prima facie, in the metric distortion framework, any decreasing function of distance $d(i,j)$ would be a natural choice for $s_{i,j}$. However, not all such functions satisfy Axiom 1. The exponential function is a popular choice in the literature employing BT or PL models. However, in general, $\frac{e^{-d(i,j)}}{e^{-d(i,j)}+e^{-d(i,j')}}\neq\frac{e^{-2d(i,j)}}{e^{-2d(i,j)}+e^{-2d(i,j')}}$, thus failing the Scale-Freeness Axiom 1.

On the other hand, observe that all functions $s=d^{-\theta}$ for any $\theta\in(0,\infty)$ satisfy our axioms. We use the regime $\theta\in(1,\infty)$ for technical simplicity in this work.

We also define the following class of functions "PQV" for $q^{\mathcal{P}(d)}(\cdot)$ motivated by Quantal Response Theory [58] and its use in probabilistic voting [20]. Observe that PQV satisfies all our axioms.

Definition 3 (Pairwise Quantal Voting (PQV))**.** *Let the relative preference $r(i,j,j')$ be the ratio of distances, $\frac{d(i,j')}{d(i,j)}$. For constant $\lambda>0$, PQV is as follows: $q^{\mathcal{P}(d)}(i,j,j')=\frac{e^{-\lambda/r(i,j,j')}}{e^{-\lambda r(i,j,j')}+e^{-\lambda/r(i,j,j')}}$.*

We now define a general class of functions for pairwise order probabilities in terms of the relative preference (ratio of distances) r. Let $\mathbf{G}$ be a class of functions such that any $\mathbf{G}\ni g:[0,\infty)\cup\{\infty\}\to[0,1]$ has the following properties.

1. g is continuous and twice-differentiable.

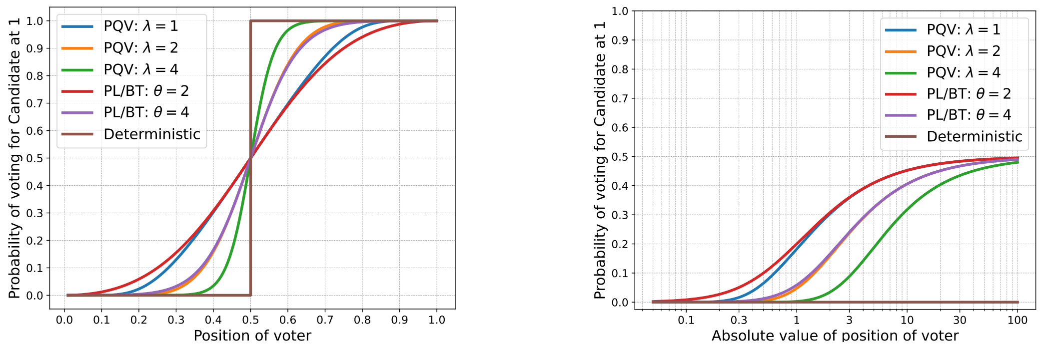

Figure 1: A 1-d Euclidean example of voting probabilities. There are two candidates at 0 and 1. The figure on the left shows the voter position between 0 and 1. In the right figure, the voter is in positions to the left of 0. As the distance grows, both candidates look similar to the voter in the probabilistic model but not in deterministic voting. The case of voter positions to the right of 1 is symmetric.

2. $g(0) = 0$. Further, $g'(r) > 0 \; \forall r \in (0, \infty)$ i.e. $g(r)$ is strictly increasing in $[0, \infty)$.
3. Define $\frac{1}{r}$ as $+\infty$ when $r = 0$. Then we must have $g(r) + g(\frac{1}{r}) = 1 \;\; \forall r \geq 0$.
4. There $\exists c \in [0, \infty)$ s.t. $g''(r) > 0 \;\; \forall r \in (0, c)$ i.e. g is convex in the open interval $(0, c)$.

Observe that PL (with $g(r) = \frac{r^\theta}{1+r^\theta}, \theta > 1$) and PQV (with $g(r) = \frac{e^{-\lambda/r}}{e^{-\lambda r}+e^{-\lambda/r}}, \lambda > 0$) are in $\mathbf{G}$. Construction of distributions (if any exists) on rankings $\sigma_\mathcal{N}$ which generate pairwise order probabilities $q^{\mathcal{P}(d)}(i, j, j') = g(\frac{d(i,j')}{d(i,j)})$ according to PQV is left for future work. We do not need it for our technical derivations. For PL, these distributions are known from prior work [40].

We assume $g \in \mathbf{G}$ in the rest of the paper. Let $\mathcal{M}(\mathcal{N} \cup \mathcal{A})$ denote the set of valid distance functions on $(\mathcal{N}, \mathcal{A})$. For any g and $d \in \mathcal{M}(\mathcal{N} \cup \mathcal{A})$ let $\hat{\mathcal{P}}^{(g)}(d)$ denote the set of probability distributions on $\sigma_\mathcal{N}$ for which the marginal pairwise order probabilities are $g(\frac{d(i,j')}{d(i,j)})$. That is,

$$\forall \mathcal{P} \in \hat{\mathcal{P}}^{(g)}(d), \sigma_\mathcal{N} \sim \mathcal{P} \implies \mathbb{P}[A \succ_i B] = g\left(\frac{d(i,B)}{d(i,A)}\right). \tag{3}$$

We assume that all voters vote independently of each other. We now define metric distortion under probabilistic voting as a function of g for a given m and n.

Definition 4 (Metric Distortion under Probabilistic Voting)**.**

$$\textsc{Dist}^{(g)}(f, n, m) := \sup_{\substack{\mathcal{N}:|\mathcal{N}|=n \\ \mathcal{A}:|\mathcal{A}|=m}} \sup_{d \in \mathcal{M}(\mathcal{N} \cup \mathcal{A})} \sup_{\mathcal{P} \in \hat{\mathcal{P}}^{(g)}(d)} \frac{\mathbb{E}_{\sigma_\mathcal{N} \sim \mathcal{P}}[SC(f(\sigma_\mathcal{N}), d)]}{\min_{A \in \mathcal{A}} SC(A, d)}. \tag{4}$$

$\textsc{dist}^{(g)}(f) = \sup_{n,m} \textsc{dist}^{(g)}(f, n, m)$ *by supremizing over all possible* n *and* m*.*

The expectation is both over the randomness in the votes and the voting rule f*.*

Observe that the distortion is a supremum over all distributions in $\hat{\mathcal{P}}^{(g)}(d)$. Since we focus on large elections (with large n and relatively small m), we define $\textsc{Dist}^{(g)}$ as a function of m and n.

As in Fig. 1, consider the 1-d Euclidean space with candidate X at the origin and Y at 1. Observe that $g\left(\frac{x}{1-x}\right)$ and $g\left(\frac{x}{1+x}\right)$ denote the probability that a voter located at a distance x from X votes for Y when the voter is to the left and right of X respectively. Interestingly, this 1-d intuition extends well for general metric spaces. Towards this, we define the following functions.

$$g_{\text{MID}}(x) := g\left(\frac{x}{1-x}\right) \forall x \in [0, 1) \text{ and } g_{\text{OUT}}(x) := g\left(\frac{x}{1+x}\right) \forall x \in [0, \infty). \tag{5}$$

Lemma 1. $\frac{g_{\text{MID}}(x)}{x}$ *and* $\frac{g_{\text{OUT}}(x)}{x}$ *have unique local maxima in* $(0, 1)$ *and* $(0, \infty)$ *respectively.*

Denote the unique maximisers of $\frac{g_{\text{MID}}(x)}{x}$ and $\frac{g_{\text{OUT}}(x)}{x}$ by x^*_{MID} and x^*_{OUT} respectively.

For simplifying notation, in the rest of the work, we use $\hat{g}_{\text{MID}}$ for $\frac{g_{\text{MID}}(x^*_{\text{MID}})}{x^*_{\text{MID}}}$ and $\hat{g}_{\text{OUT}}$ for $\frac{g_{\text{OUT}}(x^*_{\text{OUT}})}{x^*_{\text{OUT}}}$.

In the analysis in the rest of the paper, we will have $\hat{g}_{\text{MID}}$ and $\hat{g}_{\text{OUT}}$ appear many times, so we note these quantities for the PL and PQV models here. For the PL model with $\theta = 2$, $\hat{g}_{\text{MID}} = \frac{\sqrt{2}+1}{2} \approx 1.21$ and $\hat{g}_{\text{OUT}} = \frac{\sqrt{2}-1}{2} \approx 0.21$. When $\theta = 4$, $\hat{g}_{\text{MID}} \approx 1.42$ and $\hat{g}_{\text{OUT}} \approx 0.06$. When $\theta \to \infty$, $\hat{g}_{\text{MID}} \to 2$ and $\hat{g}_{\text{OUT}} \to 0$. This limit is where PL resembles deterministic voting.

For PQV with $\lambda = 1$, $\hat{g}_{\text{MID}} \approx 1.25$ and $\hat{g}_{\text{OUT}} = 0.18$. When $\lambda \to \infty$, $\hat{g}_{\text{MID}} \to 2$ and $\hat{g}_{\text{OUT}} \to 0$.

3 Distortion of Plurality Rule Under Probabilistic Voting

In this section, we give upper and lower bounds on the distortion of the Plurality rule [59] (PLU). In the limit the number of voters $n \to \infty$ ("large election"), our upper and lower bounds match and are linear in the number of candidates m. Let B represent the candidate that minimizes the social cost (referred to as 'best'), and let $\{A_j\}_{j \in [m-1]}$ denote the set of other candidates.

3.1 Upper bound on the distortion of Plurality(PLU)

Theorem 1. *For every $\epsilon > 0$ and $m \geq 2$ and $n \geq m^2$ we have*

$$\text{DIST}^{(g)}(\text{PLU}, n, m) \leq m(m-1)\left(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}}\right) \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 2m}{(2n^{(\frac{1}{2}-\epsilon)} - 1)m}\right) + \max\left(\frac{m\hat{g}_{\text{MID}}}{(1 - n^{-(\frac{1}{2}-\epsilon)})} - 1, \frac{m\hat{g}_{\text{OUT}}}{(1 - n^{-(\frac{1}{2}-\epsilon)})} + 1\right). \tag{6}$$

Further, $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{PLU}, n, m) \leq \max\left(m\hat{g}_{\text{MID}} - 1, m\hat{g}_{\text{OUT}} + 1\right)$.

To prove this theorem, we first give a lemma which upper bounds $\frac{SC(W,d)}{SC(B,d)}$ under the constraint that the expected number of voters that rank candidate W over B is given by α. This ratio will be useful to bound the contribution of non-optimal candidate W to the distortion of PLU. We state an optimization problem (7) below, which would be required to bound the ratio as a function of α.

$$\mathcal{E}_\alpha = \begin{cases} \min\limits_{\mathbf{b},\mathbf{w} \in \mathbb{R}^n_{\geq 0}} \dfrac{\sum_{i=1}^n b_i}{\sum_{i=1}^n w_i} & \\ \text{s.t.} \quad \sum\limits_{i=1}^n g\left(\dfrac{b_i}{w_i}\right) \geq \alpha & \forall \alpha \geq 0 \\ \quad \max\limits_i |w_i - b_i| \leq \min\limits_i (w_i + b_i) & \end{cases} \tag{7}$$

Lemma 2. *For any two candidates $W, B \in \mathcal{A}$ which satisfy $\sum_{i=1}^n \mathbb{P}[W \succ_i B] = \alpha$, we have*

$$\frac{SC(W,d)}{SC(B,d)} \leq \frac{1}{opt(\mathcal{E}_\alpha)} \leq \max\left(\frac{n}{\alpha}\hat{g}_{\text{MID}} - 1, \frac{n}{\alpha}\hat{g}_{\text{OUT}} + 1\right). \tag{8}$$

Our proof is via Lemmas 3 and 4. Lemma 3 shows that we can bound the ratio of social costs by the inverse of the optimum value of $\mathcal{E}_\alpha$ and Lemma 4 gives a lower bound on the optimum value of $\mathcal{E}_\alpha$.

Lemma 3. *For any two candidates $W, B \in \mathcal{A}$ satisfying $\sum_{i=1}^n \mathbb{P}[W \succ_i B] = \alpha$, we have*

$$\frac{SC(W,d)}{SC(B,d)} \leq \frac{1}{opt(\mathcal{E}_\alpha)}. \tag{9}$$

Proof. b_i and w_i in (7) represent the distances $d(i,B)$ and $d(i,W)$. The last constraint is the triangle inequality i.e. $|d(i,B) - d(i,W)| \leq d(B,W) \leq |d(i,B) + d(i,W)|$ for every voter $i \in \mathcal{N}$. □

Consider the following linearized version of (7).

$$\mathcal{E}_{\mu,\alpha} = \begin{cases} \min_{\mathbf{w},\mathbf{b}\in\mathbb{R}^n_{\geq 0}} \left(\sum_{i=1}^n b_i\right) - \mu\left(\sum_{i=1}^n w_i\right) \\ \text{s.t.} \quad \sum_{i=1}^n g\left(\frac{b_i}{w_i}\right) \geq \alpha \\ \quad |b_i - w_i| \leq 1 \; \forall i \in [n] \\ \quad b_i + w_i \geq 1 \; \forall i \in [n] \end{cases} \quad \forall 0 \leq \mu \leq 1, \alpha \geq 0. \tag{10}$$

Lemma 4. *$opt(\mathcal{E}_\alpha) \geq \min\left(\left(\frac{n}{\alpha}\hat{g}_{\text{MID}} - 1\right)^{-1}, \left(\frac{n}{\alpha}\hat{g}_{\text{OUT}} + 1\right)^{-1}\right)$.*

Our proof uses Lemma 5 and is by solving a linearized version of (7) in (10). This is done by introducing an extra non-negative parameter $\mu \leq 1$. Note that it is sufficient to consider $\mu \leq 1$ since $\text{opt}(\mathcal{E}_\alpha) \leq 1$ because B minimises the social cost by definition. We find the smallest $\mu \in (0, 1)$ such that its objective is non-negative.

Lemma 5. *If $opt(\mathcal{E}_{\mu,\alpha}) \geq 0$, then $opt(\mathcal{E}_\alpha) \geq \mu$.*

Further, $opt(\mathcal{E}_{\mu,\alpha}) \geq 0$ if $\mu = \min\left(\left(\frac{n}{\alpha}\hat{g}_{\text{MID}} - 1\right)^{-1}, \left(\frac{n}{\alpha}\hat{g}_{\text{OUT}} + 1\right)^{-1}\right)$.

The first part follows since scaling each term by a constant r satisfies the constraints and also yields the same objective. And thus we may replace the constraints by $\max_i |w_i - b_i| \leq 1$ and $\min_i(w_i + b_i) \geq 1$ in equation (10). Further, the objective function is linearized as $(\sum_{i=1}^n b_i) - \mu(\sum_{i=1}^n w_i)$.

The proof of the second part is technical and has been moved to Appendix B. It involves introducing a Lagrangian multiplier λ and demonstrating that the objective function is non-negative for a suitably chosen λ. To establish this, we show that minimising the Lagrangian over the boundaries of the constraint set given by $|b_i - w_i| = 1$ and $b_i + w_i = 1$ is sufficient. This requires a careful analysis.

The main technique used in proving Theorem 1 involves considering two cases for every non-optimal candidate A_j: one where the expected number of voters ranking candidate A_j above B (call it α_j) exceeds a threshold of $\frac{n}{m} - \frac{n^{\epsilon+1/2}}{m}$ and one where it does not. In the first case, the ratio of social costs of A_j and B is bounded using Lemma 2 that naturally gives a bound on contribution of candidate A_j to the distortion. In the later case, we use Chernoff bound to bound the probability of A_j being the winner and multiply it with the ratio of social costs of A_j and B to bound the distortion. The proof of Theorem 1 is in Appendix C.

3.2 Lower bound on the distortion of Plurality

We now present a lower bound on the distortion of PLU for any m in the limit n tends to infinity. This lower bound matches the upper bound of Theorem 1 in the limit. A full proof is in Appendix D. Note that the proof has an adversarially chosen distribution over the rankings subject to the marginals on pairwise relationships satisfying g (as in the definition of distortion under probabilistic voting 4). This lower bound does not apply to the PL model, which has a specific distribution over rankings.

Theorem 2. *For every $m \geq 2$, $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{PLU}, n, m) \geq \max\left(m\hat{g}_{\text{MID}} - 1, m\hat{g}_{\text{OUT}} + 1\right)$.*

Proof Sketch. The proof is by an example in an Euclidean metric space in $\mathbb{R}^3$. One candidate "C" is at $(1, 0, 0)$. The other $m - 1$ candidates are "good" and are equidistantly placed on a circle of radius ϵ on the $y - z$ plane centred at $(0, 0, 0)$. We call them $\mathcal{G} := \{G_1, G_2, \ldots, G_{m-1}\}$.

We present sketches of two constructions below for every $\epsilon, \zeta > 0$.

Construction 1: Let $q_{\text{MID}} := g\left(\frac{\sqrt{(x^*_{\text{MID}})^2+\epsilon^2}}{1-x^*_{\text{MID}}}\right)$ and $a_{\text{MID}} := \frac{1}{m-1}\left(1 - \frac{1+\zeta}{mq_{\text{MID}}}\right)$. Each of the $m - 1$ candidates in $\mathcal{G}$ has $\lfloor a_{\text{MID}} n \rfloor$ voters overlapping with it. The remaining voters (we call them "ambivalent") are placed at $(x^*_{\text{MID}}, 0, 0)$. Clearly, each voter overlapping with a candidate votes for it as the most preferred candidate with probability one. Each of the ambivalent voters votes as follows.

– With probability q_{MID}, vote for candidate C as the top choice and uniformly randomly permute the other candidates in the rest of the vote.

– With probability $1 - q_{\text{MID}}$, vote for candidate C as the last choice and uniformly randomly permute the other candidates in the rest of the vote.

We show that the probability that C wins tends to 1 as $n \to \infty$ and the distortion is $m\hat{g}_{\text{MID}} - 1$.

Construction 2: We give a construction where the locations of the candidates are identical as in Construction 1, and some voters are located with the "good" candidates. The ambivalent voters are at $(-x^*_{\text{OUT}}, 0, 0)$. We show that $\mathbb{P}[C \text{ wins}]$ tends to 1 as $n \to \infty$ and the distortion is $m\hat{g}_{\text{OUT}} + 1$. □

This result establishes that the distortion of Plurality is bound to increase linearly with m even under probabilistic voting, and is therefore not a good choice when m is even moderately large.

4 Distortion of Copeland Rule Under Probabilistic Voting

We now bound the distortion of the Copeland Rule. We say that candidate W defeats candidate Y if more than half of the voters rank W above Y.

Theorem 3. *For every $\epsilon > 0, m \geq 2$ and $n \geq 4$, we have*

$$\text{DIST}^{(g)}(\text{COP}, n, m) \leq 4m(m-1)\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right)(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}})^2 + \max\left(\left(\frac{2\hat{g}_{\text{MID}}}{1 - n^{-(\frac{1}{2}-\epsilon)}} - 1\right)^2, \left(\frac{2\hat{g}_{\text{OUT}}}{1 - n^{-(\frac{1}{2}-\epsilon)}} + 1\right)^2\right).$$

For every $m \geq 2$, we have $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{COP}, n, m) \leq \max\left((2\hat{g}_{\text{MID}} - 1)^2, (2\hat{g}_{\text{OUT}} + 1)^2\right)$.

Proof Sketch. A Copeland winner belongs to the uncovered set in the tournament graph, as demonstrated in [1, Theorem 15]. Recall that B denotes the candidate with the least social cost. For a Copeland winner W, either W defeats B or it defeats a candidate Y who defeats B.

We now consider two exhaustive cases on candidate A_j and define event E_j for every $j \in [m-1]$ by computing the expected fraction of votes on pairwise comparisons. The event E_j denotes the existence of an at-most two hop directed path from a candidate A_j to candidate B for Copeland such that the expected fraction of votes on all edges along that path exceed $\frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}$.

If E_j holds true, we upper bound the ratio of social cost of candidate A_j and social cost of candidate B using Lemma 2 which in-turn would give a bound on the distortion. Otherwise, we use union bound and Chernoff's bound to upper bound the probability of A_j being the winner. Multiplying the probability bound with the ratio of social costs (one obtained from Lemma 2) leads to a bound on the distortion. A detailed proof is in Appendix E. □

5 Distortion of Random Dictator Rule Under Probabilistic Voting

We first give an upper bound on the distortion of RD; the proof is in Appendix F.

Theorem 4. $\text{DIST}^{(g)}(RD, m, n) \leq (m-1)\hat{g}_{\text{MID}} + 1$.

We now give a lower bound on the distortion of RD. We do this by constructing an example.

Theorem 5. *For $m \geq 3$ and $n \geq 2$,* $\text{DIST}^{(g)}(RD, m, n) \geq 2 + \frac{1}{g^{-1}(\frac{1}{m-1})} - \frac{2}{n}$.

Proof. We have a 1-D Euclidean construction. Let B be at 0 and all other candidates $\mathcal{A} \setminus \{B\}$ be at 1. $m-1$ voters are at 0 and one voter V is at $\tilde{x} = g^{-1}(\frac{1}{m-1})/(1 + g^{-1}(\frac{1}{m-1}))$.

The ranking for V is generated as follows: pick a candidate from $\mathcal{A} \setminus \{B\}$ as the top rank uniformly at random. Keep B on the second rank. Permute the remaining candidates uniformly at random for the remaining ranks. Observe that the marginal pairwise order probabilities are consistent with the distance of V from B and each candidate in $\mathcal{A} \setminus \{B\}$. In particular $g(\frac{\tilde{x}}{1-\tilde{x}}) = \frac{1}{m-1}$. The distortion for this instance is $\mathbb{P}[B \text{ wins}] \cdot 1 + \mathbb{P}[B \text{ loses}] \cdot \frac{n-\tilde{x}}{\tilde{x}} = \frac{n-1}{n} + \frac{1}{n}\frac{n-\tilde{x}}{\tilde{x}} = 1 + \frac{1}{\tilde{x}} - \frac{2}{n} = 2 + \frac{1}{g^{-1}(\frac{1}{m-1})} - \frac{2}{n}$. $\square$

For $g(r) = \frac{r^\theta}{1+r^\theta}$, we have $g^{-1}(t) = (\frac{t}{1-t})^{\frac{1}{\theta}}$. Then $g^{-1}(\frac{1}{m-1}) = (m-2)^{-\frac{1}{\theta}}$, and the distortion lower bound is $\text{DIST}^{(g)}(\text{RD}, m, n) \geq 2 + (m-2)^{\frac{1}{\theta}} - \frac{2}{n}$, and $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{RD}, m, n) \geq 2 + (m-2)^{\frac{1}{\theta}}$. However, note that this result does not apply to the PL model! This is because the PL model has a specific distribution on the rankings. In contrast, the above result is obtained by choosing an adversarial distribution on rankings subject to the constraint that its marginals on pairwise relations are given by g. In the PL model, $\mathbb{P}[A_j \text{ is top-ranked in } \sigma_i] = \frac{d(i,A_j)^{-\theta}}{\sum_{A_k \in \mathcal{A}} d(i,A_k)^{-\theta}}$ [45]. We have the following result for the PL model. A proof via a similar construction as Theorem 5 is in Appendix G.

Theorem 6. *Let* $\text{DIST}^{\theta}_{PL}(RD, m, n)$ *denote the distortion when the voters' rankings are generated per the PL model with parameter* θ*. We have* $\lim_{n\to\infty} \text{DIST}^{\theta}_{PL}(RD, m, n) \geq 1 + \frac{(m-1)^{1/\theta}}{2}$.

6 Numerical Evaluations

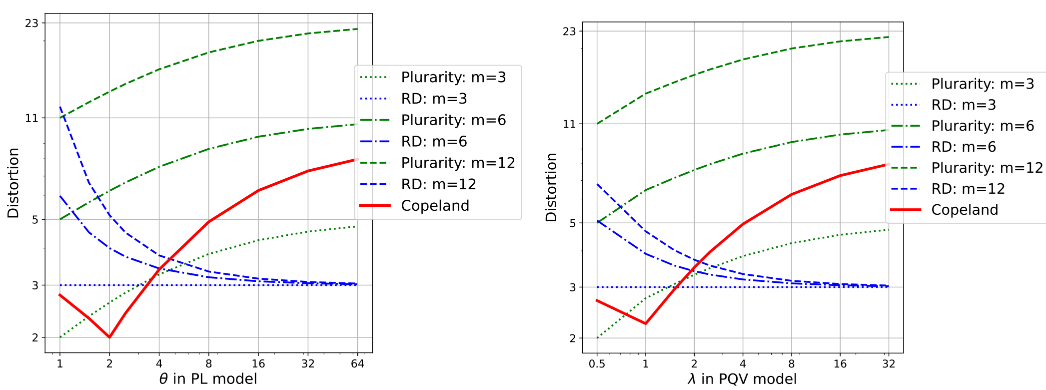

Figure 2: Here, we illustrate how the distortion bounds on different voting rules vary with m and with the randomness parameters of the two models, PL and PQV, in the limit $n \to \infty$. Both the x and y axes are on the log scale. We plot the upper bound for Copeland (Theorem 3), the lower bound for RD (Theorem 5), and the matching bounds for Plurality (Theorem 1).

Recall that higher values of θ and λ correspond to lower randomness. From Figure 2, we observe that under sufficient randomness, the more intricate voting rule Copeland outshines the simpler rule RD, which only looks at a voter's top choice. Moreover, its distortion is independent of m in the limit $n \to \infty$. This is in sharp contrast to RD, where the distortion is $\Omega(m^{1/\theta})$ in the PL model, a sharp rate of increase in m for low values of θ. The distortion of Plurality increases linearly in m.

An important observation is regarding the asymptotics when θ or λ increases. The distortion of RD converges to its value under deterministic voting, i.e., 3. The distortion of Plurality also converges to $2m - 1$, the same as in deterministic voting. Since our bound on Copeland is not tight, it converges to 9 rather than 5. So far, in the study of metric distortion, the social choice community has looked only at these asymptotic; here, we present insights available from looking at the 'complete' picture. Interestingly, the distortion of RD increases with randomness, whereas that of Copeland decreases up to a certain point and then increases again. The reason for the increases in the high randomness regime is that the votes become too noisy to reveal the best candidate any more.

Since these plots have no abrupt transitions, this figure hints that *smoothened analysis* [52] (typically done with small amounts of noise) is unlikely to give any new insights regarding metric distortion.

7 Discussion and Future Work

We extend the metric distortion framework in social choice in an important way – by capturing the bounded rationality and randomness in voters' behaviour. Consideration of this randomness shows that, in general, the original metric distortion framework is too pessimistic on important voting rules, most notably on Copeland. On the other hand, the simplistic voting rule Random Dictator, which attains a distortion of 3 (at least as good as *any* deterministic rule [1]), is not so good when we look at the full picture – its distortion increases with the number of candidates in our model. Our framework opens up opportunities to revisit the metric distortion problem with a closer-to-reality view of voters. It may hopefully lead to the development of new voting rules that consider the randomness of voters' behaviour. For example, Liu and Moitra [46] take a learning theory approach to design voting rules under the assumption of random voting per the Mallows model. However, technical analysis in our framework may be challenging because of the interplay of the *geometric* structure of voters' positions and the *probabilistic* nature of their votes.

Future Work An interesting extension would be to other tournament graph-based voting rules (weighted or unweighted). Our techniques are well-suited for this class of rules since it is based on the expected weights of the edges of the tournament graph. Closing the gap for the distortion of Copeland would be useful for getting deeper insights. Another open problem is the characterization of the set of distributions on rankings that induce the pairwise probabilities per PQV.

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A Proof of Lemma 1

Lemma (Restatement of Lemma 1). $\frac{g_{\text{MID}}(x)}{x}$ and $\frac{g_{\text{OUT}}(x)}{x}$ have unique local maxima in $(0, 1)$ and $(0, \infty)$ respectively.

To prove Lemma 1, we first state and prove Lemma 6 which shows that $g_{\text{MID}}(x)$ and $g_{\text{OUT}}(x)$ change from convex to concave in intervals $(0, 1)$ and $(0, \infty)$ respectively.

Lemma 6.
- *There* $\exists c_1 \in [0, 1]$ *s.t.* $g_{\text{MID}}(x)$ *is convex in* $[0, c_1]$ *and concave in* $[c_1, 1]$.
- *There* $\exists c_2 \in [0, \infty)$ *s.t.* $g_{\text{OUT}}(x)$ *is convex in* $[0, c_2]$ *and concave in* $[c_2, \infty)$.

Proof. Observe that $g''(x) < 0$ for $x \geq 1$.

Recall that $g_{\text{MID}}(x) = g\left(\frac{x}{1-x}\right)$ thus, $g'_{\text{MID}}(x) = g'\left(\frac{x}{1-x}\right)\frac{1}{(1-x)^2}$ and $g_{\text{MID}}(x) + g_{\text{MID}}(1-x) = 1$ Thus, $g''_{\text{MID}}(x) = g'\left(\frac{x}{1-x}\right)\frac{2}{(1-x)^3} + g''\left(\frac{x}{1-x}\right)\frac{1}{(1-x)^4}$. Observe that $g''_{\text{MID}}(0) > 0$ which implies $\lim_{x\to 1} g''_{\text{MID}}(x) < 0$ and thus, there must exist a $c \in (0, 1)$ such that $g''_{\text{MID}}(c) = 0$.

Now we show that there cannot exist two distinct $c_1, c_2 \in (0, 1)$ such that $g''_{\text{MID}}(c_1) = 0$ and $g''_{\text{MID}}(c_2) = 0$. We prove this statement by contradiction assuming the contrary which implies that $g''_{\text{MID}}(x)$ must have changed its sign twice. However, since $g'\left(\frac{x}{1-x}\right) > 0$ we must have $g''(\frac{x}{1-x})$ changing its sign twice which is a contradiction since $g''(r) > 0$ for $r \in (0, c)$ and $g''(r) < 0$ for $r \in (c, \infty)$.

Now consider $g_{\text{OUT}}(x) = g\left(\frac{x}{1+x}\right)$ we have $g'_{\text{OUT}}(x) = g'\left(\frac{x}{1+x}\right)\frac{1}{(1+x)^2}$. Thus, $g''_{\text{OUT}}(x) = -g'\left(\frac{x}{1+x}\right)\frac{2}{(1+x)^3} + g''\left(\frac{x}{1+x}\right)\frac{1}{(1+x)^4}$. Using a similar approach, we can also prove the second point in the Lemma. □

Using Lemma 6, we now prove Lemma 1 showing the existence of unique maximas of $\frac{g_{\text{MID}}(x)}{x}$ and $\frac{g_{\text{OUT}}(x)}{x}$.

Proof of Lemma 1. Recall from Lemma 6 that $g_{\text{MID}}(x)$ is convex in $[0, c_1]$ and concave in $[c_1, 1]$.

Since the first derivative equals zero at every local maxima, we must have $xg'_{\text{MID}}(x) - g(x) = 0$ for any local maxima x. We now argue that such a maxima cannot exist in $[0, c_1]$. Suppose such a maxima exists in that case, we must have $g'_{\text{MID}}(x) = \frac{g_{\text{MID}}(x) - g_{\text{MID}}(0)}{x-0}$ for some $x \in (0, c_1)$. Applying LMVT in the interval $[0, x]$[2], we must have some $t \in (0, x)$ s.t. $g'_{\text{MID}}(t) = \frac{g_{\text{MID}}(x) - g_{\text{MID}}(0)}{x-0}$, thus implying $g'_{\text{MID}}(x) = g'(t)$ contradicting the fact that $g'_{\text{MID}}(r)$ is strictly increasing in $[0, c_1]$.

Observe that $g_{\text{MID}}(t) - t\frac{g_{\text{MID}}(c_1)}{c_1}$ is zero at $t = 0$ and $t = c_1$ and thus, by Rolle's theorem, we have $g'_{\text{MID}}(x) = \frac{g_{\text{MID}}(c_1)}{c_1}$ for some $x \in (0, c_1)$. Since, $g'_{\text{MID}}(x)$ is increasing in $[0, c_1]$, we must have $g'_{\text{MID}}(c_1) > \frac{g_{\text{MID}}(c_1)}{c_1}$. Observe $\lim_{t\to 1} \frac{g_{\text{MID}}(t)}{t} = 1$ and $\frac{g_{\text{MID}}(c_1)}{c_1} > 1$. Also, we have $\frac{d}{dt}\left(\frac{g_{\text{MID}}(t)}{t}\right)\Big|_{t=c_1} > 0$ since $c_1 g'_{\text{MID}}(c_1) > g_{\text{MID}}(c_1)$ implying $g_{\text{MID}}(t)/t$ is increasing at $t = c_1$. Thus, $g_{\text{MID}}(t)/t$ must have at least one local maxima x^* in the open interval (c_1, ∞) and no local maxima elsewhere.

We now argue that this local maxima x^* is unique. Suppose we have two distinct local maximas at $x_1, x_2 \in (c_1, \infty)$ and thus, we have $x_1 g'_{\text{MID}}(x_1) - g_{\text{MID}}(x_1) = 0$ and $x_2 g'_{\text{MID}}(x_2) - g_{\text{MID}}(x_2) = 0$. Rolle's theorem would imply that there exists $t \in (x_1, x_2)$[3] s.t. $tg''_{\text{MID}}(t) = 0$ which is a contradiction since $g''_{\text{MID}}(x) < 0$ in (c_1, ∞).

Similarly, we can prove the result on the existence and uniqueness of maxima of the function $\frac{g\left(\frac{x}{x+1}\right)}{x}$. □

B Proof of Lemma 5

Lemma (Restatement of Lemma 5). If $\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq 0$, then $\text{opt}(\mathcal{E}_{\alpha}) \geq \mu$.

Further, $\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq 0$ if $\mu = \min\left(\left(\frac{n}{\alpha}\hat{g}_{\text{MID}} - 1\right)^{-1}, \left(\frac{n}{\alpha}\hat{g}_{\text{OUT}} + 1\right)^{-1}\right)$.

[2] Observe that $g(x)/x$ has a removable discontinuity at 0 since the limit is defined.

[3] W.L.O.G, we assume $x_1 < x_2$

Proof. To lower bound the optimal value of $\mathcal{E}_{\mu,\alpha}$, we first pre-multiply the first constraint by λ (and substitute $\frac{b_i}{w_i} = r_i \ \forall i \in [n]$) and thus define,

$$F(\mathbf{r}, \mathbf{b}, \lambda) = \left(\sum_{i=1}^{n} b_i\right) - \mu \left(\sum_{i=1}^{n} \frac{b_i}{r_i}\right) - \lambda \left(\sum_{i=1}^{n} g(r_i) - \alpha\right). \tag{11}$$

Further, we define the set which satisfies the last two constraints in $\mathcal{E}_{\mu,\alpha}$ by $\mathcal{C}$ as

$$\mathcal{C} := \{(\mathbf{r}, \mathbf{b}) \in (\mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{\geq 0}) : b_i(1 + 1/r_i) \geq 1; |b_i(1/r_i - 1)| \leq 1 \ \forall i \in [n]\}. \tag{12}$$

From the theory of Lagrangian, we have the following

$$\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq \min_{(\mathbf{r},\mathbf{b}) \in \mathcal{C}} \max_{\lambda \geq 0} F(\mathbf{r}, \mathbf{b}, \lambda) \geq \max_{\lambda \geq 0} \min_{(\mathbf{r},\mathbf{b}) \in \mathcal{C}} F(\mathbf{r}, \mathbf{b}, \lambda). \tag{13}$$

Now for a fixed $\lambda > 0$, we minimise $F(\mathbf{r}, \mathbf{b}, \lambda)$ over $(\mathbf{r}, \mathbf{b}) \in \mathcal{C}$. Observe that for every $i \in [n]$, it is sufficient to minimise $h(r_i, b_i)$ defined as follows.

$$h(r_i, b_i) := b_i(1 - \mu/r_i) - \lambda \left(g(r_i) - \frac{\alpha}{n}\right). \tag{14}$$

Observe that the constraints in $\mathcal{C}$ can be written as $b_i \geq \frac{r_i}{1+r_i}$ and $b_i \leq \frac{r_i}{|1-r_i|}$.

Observe that for a given r_i, the function $h(r_i, b_i)$ is monotonic in b_i and thus the optimum point must lie on the boundary and first optimize over $b_i(1 + 1/r_i) = 1$ (call it $\mathcal{C}_i^{\text{MID}}$) and $|b_i(1 - 1/r_i)| = 1$ (call it $\mathcal{C}_i^{\text{OUT}}$) respectively.

Recall from Lemma 6 that there exists c_1, c_2 s.t. $g_{\text{MID}}(x)$ is convex in $(0, c_1)$ and concave in $(c_1, 1)$ and $g_{\text{OUT}}(x)$ is convex in $(0, c_2)$ and concave in (c_2, ∞).

- *Minimisation of* $h(r_i, b_i)$ *over* $b_i(1 + 1/r_i) = 1$.

  We first substitute $1/r_i = 1/b_i - 1$ in the function and thus, can write the function $h(b_i) = b_i(\mu + 1) - \mu - \lambda \left(g\left(\frac{b_i}{1-b_i}\right) - \frac{\alpha}{n}\right) = b_i(\mu + 1) - \mu - \lambda \left(g_{\text{MID}}(b_i) - \frac{\alpha}{n}\right)$.

  Observe that on optimizing over b_i, we obtain two local minima, one at $b_i = 0$ and the other at $b_i = \tilde{x}^{\text{MID}}(\lambda) \in (c_1, \infty)$ where $\tilde{x}^{\text{MID}}(\lambda)$ satisfies the following equations if $\lambda \geq \frac{1+\mu}{g'_{\text{MID}}(c_1)}$. Otherwise, we have a unique minima at $b_i = 0$. [4]

  $$g'_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda)) = \max\left(\frac{1 + \mu}{\lambda}, g'_{\text{MID}}(1^-)\right) \text{ and } g''_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda)) < 0. \tag{15}$$

  Observe $\tilde{x}^{\text{MID}}(\lambda) > c_1$ since g_{MID} is concave only in $[c_1, 1]$. Also observe that since $g'_{\text{MID}}(x)$ is monotonically increasing, $\tilde{x}^{\text{MID}}(\lambda)$ is monotonically increasing in λ.

- *Minimisation of* $h(r_i, b_i)$ *over* $b_i|(1 - 1/r_i)| = 1$.

  On substituting, we write the function

  $$h(b_i) = \begin{cases} (1-\mu)b_i - \mu - \lambda\left(g\left(\frac{b_i}{1+b_i}\right) - \frac{\alpha}{n}\right) = (1-\mu)b_i - \mu - \lambda\left(g_{\text{OUT}}(b_i) - \frac{\alpha}{n}\right) & \text{if } r_i \geq 1 \\ (1-\mu)b_i + \mu - \lambda\left(g\left(\frac{b_i}{b_i-1}\right) - \frac{\alpha}{n}\right) \overset{(a)}{=} (1-\mu)b_i + \mu - \lambda + \lambda\left(g_{\text{OUT}}(b_i - 1) + \frac{\alpha}{n}\right) & \text{otherwise} \end{cases} \tag{16}$$

  (a) follows from the fact that $g(r) + g(1/r) = 1$.

  Since the second function has only a single minima at $b_i = 1$, it is sufficient to consider only the first function in the case $r_i \geq 1$.

  Observe that on optimizing over b_i, we obtain two local minima one at $b_i = 0$ and one at $b_i = \tilde{x}^{\text{OUT}}(\lambda) \in (c_2, \infty)$ where $\tilde{x}^{\text{OUT}}(\lambda)$ satisfies the equations if $\lambda \geq \frac{1-\mu}{g'_{\text{OUT}}(c_2)}$. Otherwise, we have a unique minima at $b_i = 0$. [5]

  $$g'_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda)) = \left(\frac{1 - \mu}{\lambda}\right) \text{ and } g''_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda)) < 0. \tag{17}$$

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[4] This follows from the fact that $g_{\text{MID}}(x)$ is monotonically decreasing in $[c_1, 1)$ and monotonically increasing in $[0, c_1)$.

[5] This follows from the fact that $g_{\text{OUT}}(x)$ is monotonically decreasing in $[c_2, \infty)$ and monotonically increasing in $[0, c_2)$. Since $g'_{\text{OUT}}(\infty) = 0$, the solution to (17) exists for every $\lambda \in \left(\frac{1-\mu}{g'_{\text{OUT}}(c_2)}, \infty\right)$.

Thus, we have $\tilde{x}^{\text{OUT}}(\lambda) > c_2$ since g_{OUT} is concave only in $[c_2, \infty)$. Also observe that since $g'_{\text{OUT}}(x)$ is monotonic, $\tilde{x}^{\text{OUT}}(\lambda)$ is monotonic in λ.

Since, this argument is true for every $i \in [n]$, we obtain

$$\min_{(\mathbf{r},\mathbf{b})} F(\mathbf{r}, \mathbf{b}, \lambda) = n \cdot \min\Bigg(-\mu + \lambda\frac{\alpha}{n}, (\mu+1)\tilde{x}^{\text{MID}}(\lambda) - \mu - \lambda\left(g_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda)) - \frac{\alpha}{n}\right), \\ (1-\mu)\tilde{x}^{\text{OUT}}(\lambda) - \mu - \lambda\left(g_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda)) - \frac{\alpha}{n}\right)\Bigg). \tag{18}$$

Since x^*_{MID} is the local maximiser of $\frac{g_{\text{MID}}(x)}{x}$, we have

$$g_{\text{MID}}(x^*_{\text{MID}}) = x^*_{\text{MID}}\hat{g}_{\text{MID}} \text{ and } x^*_{\text{MID}} > c_1. \tag{19}$$

Similarly,

$$g_{\text{OUT}}(x^*_{\text{OUT}}) = x^*_{\text{OUT}}\hat{g}_{\text{OUT}} \text{ and } x^*_{\text{OUT}} > c_2. \tag{20}$$

For the purpose of this analysis, we define two functions $\delta^{\text{MID}}(\lambda)$ and $\delta^{\text{OUT}}(\lambda)$ below.

$$\delta^{\text{MID}}(\lambda) = (\mu+1)\tilde{x}^{\text{MID}}(\lambda) - \mu - \lambda\left(g_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda)) - \frac{\alpha}{n}\right). \tag{21}$$

$$\delta^{\text{OUT}}(\lambda) = (1-\mu)\tilde{x}^{\text{OUT}}(\lambda) - \mu - \lambda\left(g_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda)) - \frac{\alpha}{n}\right). \tag{22}$$

We also define

$$\mu^* := \min\left(\left(\frac{n}{\alpha}\hat{g}_{\text{MID}} - 1\right)^{-1}, \left(\frac{n}{\alpha}\hat{g}_{\text{OUT}} + 1\right)^{-1}\right), \quad \text{and} \quad \lambda^* := \mu^*\frac{n}{\alpha}. \tag{23}$$

Recall that we aim to show $\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq 0$ when $\mu = \mu^*$ and thus substitute $\mu = \mu^*$ in every subsequent equation. Observe that it is sufficient to show $\delta^{\text{MID}}(\lambda^*)$ and $\delta^{\text{OUT}}(\lambda^*)$ are non-negative since this would imply that $\max_{\lambda \geq 0} \min_{(\mathbf{r},\mathbf{b}) \in \mathcal{C}} F(\mathbf{r}, \mathbf{b}, \lambda)$ is non-negative.

We now consider the following two exhaustive cases.

- Case 1: $\hat{g}_{\text{MID}} - \frac{\alpha}{n} > \hat{g}_{\text{OUT}} + \frac{\alpha}{n}$. Observe from Equation (15),

$$g'_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda^*)) = \max\left(\frac{n}{\alpha}\left(\frac{1}{\mu^*} + 1\right), g'_{\text{MID}}(1^-)\right) = g'_{\text{MID}}(x^*_{\text{MID}}) \overset{(d)}{\implies} \tilde{x}^{\text{MID}}(\lambda^*) = x^*_{\text{MID}}. \tag{24}$$

(d) follows from the fact that both $\tilde{x}^{\text{MID}}(\lambda^*)$ and x^*_{MID} exceed c_1 and $g'_{\text{MID}}(x)$ is monotonically decreasing for $x \geq c_1$.

$$\delta^{\text{MID}}(\lambda^*) = (\mu^*+1)\tilde{x}^{\text{MID}}(\lambda^*) - \mu^* - \lambda^*\left(g_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda^*)) - \frac{\alpha}{n}\right) \\ \overset{(b)}{\geq} (-\mu^* + \lambda^*\alpha/n) + \lambda^*\left(\tilde{x}^{\text{MID}}(\lambda^*)g'_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda^*)) - g_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda^*))\right) \overset{(c)}{\geq} 0. \tag{25}$$

(b) follows from $g'_{\text{MID}}(\tilde{x}^{\text{MID}}(\lambda^*)) = \frac{1+\mu^*}{\lambda^*}$ [6] as stated in Equation (15).

(c) follows from $\tilde{x}^{\text{MID}}(\lambda^*) = x^*_{\text{MID}}$ (in Equation (24)) and $g_{\text{MID}}(x^*_{\text{MID}}) = x^*_{\text{MID}}\hat{g}_{\text{MID}}$ (in Equation (19)) and the fact that $\mu^* = \lambda^*\frac{\alpha}{n}$. Now consider,

$$\hat{g}_{\text{OUT}} \overset{(d)}{=} \frac{g_{\text{OUT}}(x^*_{\text{OUT}})}{x^*_{\text{OUT}}} \overset{(e)}{\leq} \frac{g_{\text{MID}}(x^*_{\text{MID}})}{x^*_{\text{MID}}} - 2\alpha/n \overset{(g)}{=} \frac{1-\mu}{\lambda^*} \overset{(h)}{\implies} g'_{\text{OUT}}(x^*_{\text{OUT}}) \leq g'_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*)) \overset{(i)}{\implies} x^*_{\text{OUT}} \geq \tilde{x}^{\text{OUT}}(\lambda^*).$$

(d) follows from the fact that x^*_{OUT} is the local maximiser of $g_{\text{OUT}}(x)/x$,

(e) follows from the fact that $\hat{g}_{\text{MID}} - \frac{\alpha}{n} > \hat{g}_{\text{OUT}} + \frac{\alpha}{n}$ in Case 1.

(g) follows from the definition of λ^* and that $\mu = \lambda^*\frac{\alpha}{n}$.

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[6] This follows from the fact that $\frac{1+\mu^*}{\lambda^*} = \hat{g}_{\text{MID}} = g'(x^*_{\text{MID}}) \geq g'_{\text{MID}}(1^-)$

(h) follows from the constraint in (17).

(i) follows from the fact that $g'_{\text{OUT}}(x)$ is monotonically decreasing in x in $[c_2, \infty)$.

$$\begin{aligned}\delta^{\text{OUT}}(\lambda^*) &= (1-\mu)\tilde{x}^{\text{OUT}}(\lambda^*) - \mu - \lambda^* \left(g_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*)) - \frac{\alpha}{n}\right) \\ &\overset{(j)}{=} \left(-\mu + \lambda^* \frac{\alpha}{n}\right) + \lambda^*(\tilde{x}^{\text{OUT}}(\lambda^*) g'_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*)) - g_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*))) \\ &\overset{(k)}{\geq} 0 + 0 \geq 0. \end{aligned} \tag{26}$$

(j) follows from $g'_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda)) = \frac{1-\mu}{\lambda}$ as stated in Equation (17), and

(k) follows from the following reasons:

- Observe that $xg'_{\text{OUT}}(x) - g_{\text{OUT}}(x)$ is monotonically decreasing in $[c_2, \infty)$ as g_{OUT} is concave in this region. However, since $x^*_{\text{OUT}} \geq \tilde{x}^{\text{OUT}}(\lambda^*) \geq c_2$, we have
$$(\tilde{x}^{\text{OUT}}(\lambda^*) g'_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*)) - g_{\text{OUT}}(\tilde{x}^{\text{OUT}}(\lambda^*))) \geq x^*_{\text{OUT}}\hat{g}_{\text{OUT}} - g_{\text{OUT}}(x^*_{\text{OUT}}) = 0$$
- $\lambda^* = \mu^* \frac{n}{\alpha}$ follows from the definition of λ^*.

Thus, using (25) and (26) we show that for the chosen value of $\lambda^* = \mu^* \frac{n}{\alpha}$, we have $\min_{(\mathbf{r},\mathbf{w})\in\mathcal{C}} F(\mathbf{r}, \mathbf{w}, \lambda^*) \geq 0$ implying from (13) that $\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq 0$.

- Case 2: $\hat{g}_{\text{MID}} - \frac{\alpha}{n} \leq \hat{g}_{\text{OUT}} + \frac{\alpha}{n}$
Choosing $\lambda^* = \mu^* \frac{n}{\alpha}$, we can prove $\text{opt}(\mathcal{E}_{\mu,\alpha}) \geq 0$ in a very similar manner whenever $\mu = \mu^*$. ☐

C Proof of Theorem 1

Theorem (Restatement of Theorem 1). For every $\epsilon > 0$ and $m \geq 2$ and $n \geq m^2$ we have

$$\begin{aligned}\text{DIST}^{(g)}(\text{PLU}, n, m) \leq\ & m(m-1)\left(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}}\right) \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 2m}{(2n^{(\frac{1}{2}-\epsilon)} - 1)m}\right) \\ &+ \max\left(\frac{m\hat{g}_{\text{MID}}}{(1-n^{-(\frac{1}{2}-\epsilon)})} - 1, \frac{m\hat{g}_{\text{OUT}}}{(1-n^{-(\frac{1}{2}-\epsilon)})} + 1\right). \end{aligned} \tag{27}$$

Further, $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{PLU}, n, m) \leq \max\left(m\hat{g}_{\text{MID}} - 1, m\hat{g}_{\text{OUT}} + 1\right).$

Proof. Recall that candidate $B \in \mathcal{A}$ minimises the social cost. The other candidates are denoted by $\{A_j\}_{j\in[m-1]}$.

$$\text{DIST}^{(g)}(\text{PLU}, n, m) = \sup_{d\in\mathcal{M}(\mathcal{N}\cup\mathcal{A})} \left(\sum_{j=1}^{m-1} \mathbb{P}[A_j \text{ wins}] \frac{\text{SC}(A_j, d)}{\text{SC}(B, d)} + \mathbb{P}[B \text{ wins}]\right) \tag{28}$$

For every $j \in [m-1]$, we now bound the probability of A_j being the winner. This event implies that at least $\frac{n}{m}$ voters choose A_j as the top preference, implying that the same voters rank A_j over B. Further, we now define Bernoulli random variables $\{Y_{i,j}\}_{i=1}^n$ each denoting the event that voter i ranks candidate A_j over B. Recall from Equation 3, $Y_{i,j} \sim \text{Bern}\left(g\left(\frac{d(i,B)}{d(i,A_j)}\right)\right)$. Therefore,

$$\mathbb{P}[A_j \text{ wins}] \leq \mathbb{P}\left(\sum_{i=1}^n Y_{i,j} \geq \frac{n}{m}\right). \tag{29}$$

Let α_j be the expectation of the random variable $\sum_{i=1}^n Y_{i,j}$ i.e. the expected number of voters ranking A_j over B.

$$\alpha_j := \sum_{i=1}^n \mathbb{E}[Y_{i,j}] = \sum_{i=1}^n g\left(\frac{d(i,B)}{d(i,A_j)}\right) \text{ for every } j \in [m-1]. \tag{30}$$

Now we use Chernoff bounds on the sum of Bernoulli random variable for every $j \in [m-1]$ when $\alpha_j \leq \frac{n}{m} - \frac{n^{(1/2+\epsilon)}}{m}$ to bound the probability of A_j being the winner.

If $\alpha_j \leq \dfrac{n}{m} - \dfrac{n^{(1/2+\epsilon)}}{m}$ we have,

$$\mathbb{P}[A_j \text{ wins}] \leq \mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \geq \frac{n}{m}\right) = \mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \geq \alpha_j \left(1 + \frac{n}{m\alpha_j} - 1\right)\right) \tag{31}$$

$$\stackrel{(a)}{\leq} \left(\frac{e^{(\frac{n}{m\alpha_j}-1)}}{(\frac{n}{m\alpha_j})^{n/m\alpha_j}}\right)^{\alpha_j} \tag{32}$$

$$= \left(\frac{m\alpha_j}{n}\right)^{\frac{n}{m}} e^{\frac{n}{m}-\alpha_j} \tag{33}$$

$$\leq \frac{m\alpha_j}{n}\left(\frac{m\alpha_j}{n}\exp\left(-\frac{\alpha_j}{n/m-1}\right)\right)^{(\frac{n}{m}-1)} e^{\frac{n}{m}} \tag{34}$$

$$\stackrel{(c)}{\leq} \frac{m\alpha_j}{n} e^{\frac{n}{m}} \left(\left(1 - n^{-(\frac{1}{2}-\epsilon)}\right)\exp\left(-\frac{\frac{n}{m} - \frac{n^{(\frac{1}{2}+\epsilon)}}{m}}{n/m-1}\right)\right)^{\frac{n}{m}-1} \tag{35}$$

$$= \frac{m\alpha_j}{n}\left(1 - n^{-(\frac{1}{2}-\epsilon)}\right)^{(n/m-1)} \exp\left(\frac{n^{(\frac{1}{2}+\epsilon)}}{m}\right) \tag{36}$$

$$\stackrel{(d)}{\leq} \frac{m\alpha_j}{n}\exp\left(\frac{-2n^{-(\frac{1}{2}-\epsilon)}(n/m-1)}{2 - n^{-(\frac{1}{2}-\epsilon)}} + \frac{n^{(\frac{1}{2}+\epsilon)}}{m}\right) \tag{37}$$

$$= \frac{m\alpha_j}{n}\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 2m}{(2n^{(\frac{1}{2}-\epsilon)} - 1)m}\right). \tag{38}$$

(a) follows from applying the Chernoff bound. We restate the bound from [60] below.

Suppose $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{P}(X_i) = \mu_i$ for every $i \in [n]$ and $\mu := \sum_{i=1}^{n} \mu_i$, then we have

$$\mathbb{P}(\sum_i X_i \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \tag{39}$$

(c) holds since xe^{-x} is increasing in $(0,1)$ and because $\frac{\alpha}{n/m-1} \leq 1$ and $\alpha \leq \frac{n}{m} - \frac{n^{(\frac{1}{2}+\epsilon)}}{m}$, the maxima is attained at $\alpha = \frac{n}{m} - \frac{n^{(\frac{1}{2}+\epsilon)}}{m}$. (d) holds since $\log(1+x) \leq \frac{2x}{2+x}$ for $-1 < x \leq 0$.

Let $S := \{j \in [m-1] : \alpha_j < \frac{n}{m} - \frac{n^{(1/2+\epsilon)}}{m}\}$ i.e. S denotes the indices of candidates with α_j less than $\frac{n}{m} - \frac{n^{(1/2+\epsilon)}}{m}$.

Now using Lemma 2 and $\alpha_j \geq \frac{n}{m} - \frac{n^{(1/2+\epsilon)}}{m}$ for every $j \in [m-1] \setminus S$, we have

$$\frac{SC(A_j, d)}{SC(B, d)} \leq \max\left(\frac{m\hat{g}_{\text{MID}}}{(1 - n^{-(1/2-\epsilon)})} - 1, \frac{m\hat{g}_{\text{OUT}}}{(1 - n^{-(1/2-\epsilon)})} + 1\right) \tag{40}$$

We now have

$$\begin{aligned}
&\textsc{dist}^{(g)}(\text{PLU}, n, m)\\
&= \sup_{d \in \mathcal{M}(\mathcal{N}\cup\mathcal{A})}\left(\sum_{j \in [m-1]\setminus S}\left(\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j, d)}{\text{SC}(B, d)}\right) + \mathbb{P}[B \text{ wins}] + \sum_{j \in S}\left(\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j, d)}{\text{SC}(B, d)}\right)\right)\\
&\stackrel{(a)}{\leq} \max\left(\max_{j \in [m-1]\setminus S}\frac{SC(A_j, d)}{SC(B, d)}, 1\right) + \sum_{j \in S}\left(\max\left(\frac{n}{\alpha_j}\hat{g}_{\text{MID}} - 1, \frac{n}{\alpha_j}\hat{g}_{\text{OUT}} + 1\right)\frac{m\alpha_j}{n}\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 2m}{(2n^{(\frac{1}{2}-\epsilon)} - 1)m}\right)\right)\\
&\stackrel{(b)}{\leq} m(m-1)\left(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}}\right)\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 2m}{(2n^{(\frac{1}{2}-\epsilon)} - 1)m}\right) + \max\left(\frac{m\hat{g}_{\text{MID}}}{(1 - n^{-(1/2-\epsilon)})} - 1, \frac{m\hat{g}_{\text{OUT}}}{(1 - n^{-(1/2-\epsilon)})} + 1\right).
\end{aligned}$$

(a) follows from the following observations.

- Apply Lemma 2 to bound $\frac{SC(A_j,d)}{SC(B,d)}$. Since $\alpha_j \leq \frac{n}{m} - \frac{n^{(1/2+\epsilon)}}{m}$ $\forall j \in S$, apply Equation (31) to bound $\mathbb{P}[A_j \text{ wins}]$.
- $\sum_{j\in[m-1]\setminus S} \left(\mathbb{P}[A_j \text{ wins}] \frac{\mathrm{SC}(A_j,d)}{\mathrm{SC}(B,d)}\right) + \mathbb{P}[B \text{ wins}] \leq \max\left(\max_{j\in[m-1]\setminus S} \frac{SC(A_j,d)}{SC(B,d)}, 1\right)$.

(b) follows from the fact that $|S| \leq m-1$, $\max(a,b) \leq a+b$, and applying Equation (40). □

D Proof of Theorem 2

Theorem (Restatement of Theorem 2). For every $m \geq 2$, $\lim_{n\to\infty} \mathrm{DIST}^{(g)}(\mathrm{PLU}, n, m) \geq \max(m\hat{g}_{\mathrm{MID}} - 1, m\hat{g}_{\mathrm{OUT}} + 1)$.

Proof. The proof is by an example in an Euclidean metric space in $\mathbb{R}^3$. One candidate "C" is at $(1,0,0)$. The other $m-1$ candidates are "good" and are equidistantly placed on a circle of radius ϵ on the $y-z$ plane centred at $(0,0,0)$. We call them $\mathcal{G} := \{G_1, G_2, \ldots, G_{m-1}\}$.

We present two constructions below for every $\epsilon, \zeta > 0$.

Construction 1: Let $q_{\mathrm{MID}} := g\left(\frac{\sqrt{(x^*_{\mathrm{MID}})^2+\epsilon^2}}{1-x^*_{\mathrm{MID}}}\right)$ and $a_{\mathrm{MID}} := \frac{1}{m-1}\left(1 - \frac{1+\zeta}{mq_{\mathrm{MID}}}\right)$. Each of the $m-1$ candidates in $\mathcal{G}$ has $\lfloor a_{\mathrm{MID}} n \rfloor$ voters overlapping with it. The remaining voters (we call them "ambivalent") are placed at $(x^*_{\mathrm{MID}}, 0, 0)$. Clearly, each voter overlapping with a candidate votes for it as the most preferred candidate with probability one. Each of the ambivalent voters votes as follows.

– With probability q_{MID}, vote for candidate C as the top choice and uniformly randomly permute the other candidates in the rest of the vote.

– With probability $1 - q_{\mathrm{MID}}$, vote for candidate C as the last choice and uniformly randomly permute the other candidates in the rest of the vote.

Observe that this satisfies the pairwise probability criterion in Equation 3. Since $\lim_{n\to\infty} \lfloor an \rfloor / n = a$ and that the distance of a candidate in $\mathcal{G}$ from any non-ambivalent voter is at most 2ϵ, we have that for every $j \in [m-1]$,

$$\lim_{n\to\infty} \frac{SC(C,d)}{SC(G_j,d)} \geq \frac{(1-x^*_{\mathrm{MID}})(1-(m-1)a_{\mathrm{MID}}) + (m-1)a_{\mathrm{MID}}\sqrt{1+\epsilon^2}}{(1-(m-1)a_{\mathrm{MID}})\sqrt{(x^*_{\mathrm{MID}})^2+\epsilon^2} + 2(m-2)a_{\mathrm{MID}}\epsilon} \tag{41}$$

$$= \frac{(mq_{\mathrm{MID}} - (1+\zeta))\sqrt{1+\epsilon^2} + (1+\zeta)(1-x^*_{\mathrm{MID}})}{(1+\zeta)\sqrt{(x^*_{\mathrm{MID}})^2+\epsilon^2} + 2(m-2)a_{\mathrm{MID}}\epsilon}. \tag{42}$$

Clearly every candidate in $\mathcal{G}$ minimises the social cost and now we show that $\lim_{n\to\infty} \mathbb{P}[C \text{ wins}] = 1$.

Let Bernoulli random variables $\{Y_i\}_{i=1}^n$ denote the events that voter $i \in \mathcal{N}$ ranks candidate C at the top. Here, $\sum_{i=1}^n \mathbb{P}[Y_i = 1] = q_{\mathrm{MID}}(n - (m-1)\lfloor a_{\mathrm{MID}} n \rfloor)$ and thus

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{P}[Y_i = 1]}{n} = \frac{1+\zeta}{m}.$$

By the law of large numbers, we have that $\mathbb{P}[\sum_i Y_i \geq \frac{n}{m}] = 1$ as $n \to \infty$. Since every candidate in $\mathcal{G}$ is equally likely to win, the event $\sum_i Y_i \geq \frac{n}{m}$ implies the event that C is the winner and thus, $\lim_{n\to\infty} \mathbb{P}[C \text{ wins}] = 1$. Thus,

$$\lim_{n\to\infty} \mathrm{DIST}^{(g)}(\mathrm{PLU}, n, m) \geq \frac{(mq_{\mathrm{MID}} - (1+\zeta))\sqrt{1+\epsilon^2} + (1+\zeta)(1-x^*_{\mathrm{MID}})}{(1+\zeta)\sqrt{(x^*_{\mathrm{MID}})^2+\epsilon^2} + 2(m-2)a_{\mathrm{MID}}\epsilon}. \tag{43}$$

Construction 2: Let $q_{\mathrm{OUT}} := g\left(\frac{\sqrt{(x^*_{\mathrm{OUT}})^2+\epsilon^2}}{1+x^*_{\mathrm{OUT}}}\right)$ and $a_{\mathrm{OUT}} := \frac{1}{m-1}\left(1 - \frac{1+\zeta}{mq_{\mathrm{OUT}}}\right)$. Each candidate in $\mathcal{G}$ has $\lfloor a_{\mathrm{OUT}} n \rfloor$ voters overlapping with it, and the remaining "ambivalent" voters are at $(-x^*_{\mathrm{OUT}}, 0, 0)$.

Clearly, each voter overlapping with a candidate votes for it as the most preferred candidate with probability one. Each of the ambivalent voters votes as follows.

- With probability q_{OUT}, vote for candidate C as the top choice and uniformly randomly permute the other candidates in the rest of the vote.
- With probability $1 - q_{\text{OUT}}$, vote for candidate C as the last choice and uniformly randomly permute the other candidates in the rest of the vote.

This satisfies the pairwise probability criterion in Equation 3. For every $j \in [m-1]$,

$$\lim_{n\to\infty} \frac{SC(C,d)}{SC(G_j,d)} \geq \frac{(1+x^*_{\text{OUT}})(1-(m-1)a_{\text{OUT}}) + (m-1)a_{\text{OUT}}\sqrt{1+\epsilon^2}}{(1-(m-1)a_{\text{OUT}})\sqrt{(x^*_{\text{OUT}})^2+\epsilon^2} + 2(m-2)a_{\text{OUT}}\epsilon} \tag{44}$$

$$= \frac{(1+\zeta)(1+x^*_{\text{OUT}}) + (mq_{\text{OUT}} - (1+\zeta))\sqrt{1+\epsilon^2}}{(1+\zeta)\sqrt{(x^*_{\text{OUT}})^2+\epsilon^2} + 2(m-2)a_{\text{OUT}}\epsilon}. \tag{45}$$

Clearly, every candidate in $\mathcal{G}$ minimises the social cost. Now, we show that $\lim_{n\to\infty} \mathbb{P}[C \text{ wins}] = 1$.

Let Bernoulli random variables $\{Y_i\}_{i=1}^n$ denote the events that voter $i \in \mathcal{N}$ ranks candidate C at the top. We have $\sum_{i=1}^n \mathbb{P}[Y_i = 1] = q_{\text{MID}}(n-(m-1)\lfloor an \rfloor)$ and thus, $\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{P}[Y_i=1]}{n} = \frac{1+\zeta}{m}$. Applying the law of large numbers, we get that $\mathbb{P}[\sum_i Y_i \geq \frac{n}{m}] = 1$ as n tends to ∞. However since every candidate in $\mathcal{G}$ is equally likely to win, the event $\sum_i Y_i \geq \frac{n}{m}$ corresponds to the event that C is the winner and thus, $\lim_{n\to\infty} \mathbb{P}[C \text{ wins}] = 1$. Therefore we have,

$$\lim_{n\to\infty} \text{DIST}^{(g)}(\text{PLU}, n, m) \geq \frac{(mq_{\text{OUT}} - (1+\zeta))\sqrt{1+\epsilon^2} + (1+\zeta)(1+x^*_{\text{OUT}})}{(1+\zeta)\sqrt{(x^*_{\text{OUT}})^2+\epsilon^2} + 2(m-2)a_{\text{OUT}}\epsilon}. \tag{46}$$

On applying the limit $\epsilon, \zeta \to 0$ and substituting for q_{MID} and q_{OUT}, we get the desired lower bound by combining the results from the two constructions. □

E Proof of Theorem 3

Theorem 7. *Restatement of Theorem 3 For every $\epsilon > 0, m \geq 2$ and $n \geq 4$, we have*

$$\text{DIST}^{(g)}(\text{COP}, n, m) \leq 4m(m-1)\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right)(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}})^2$$
$$+ \max\Big(\Big(\frac{2\hat{g}_{\text{MID}}}{1-n^{-(\frac{1}{2}-\epsilon)}} - 1\Big)^2, \Big(\frac{2\hat{g}_{\text{OUT}}}{1-n^{-(\frac{1}{2}-\epsilon)}} + 1\Big)^2\Big).$$

For every $m \geq 2$, we have $\lim_{n\to\infty} \text{DIST}^{(g)}(\text{COP}, n, m) \leq \max\left((2\hat{g}_{\text{MID}} - 1)^2, (2\hat{g}_{\text{OUT}} + 1)^2\right)$.

Proof. Recall that $B \in \mathcal{A}$ minimises the social cost, and $\{A_j\}_{j\in[m-1]}$ denotes the set $\mathcal{A} \setminus B$.

$$\text{DIST}^{(g)}(\text{COP}, n, m) = \sup_{d\in\mathcal{M}(\mathcal{N}\cup\mathcal{A})} \left(\sum_{j=1}^{m-1} \mathbb{P}[A_j \text{ wins}] \frac{\text{SC}(A_j, d)}{\text{SC}(B, d)} + \mathbb{P}[B \text{ wins}]\right) \tag{47}$$

Consider a Copeland winner W. As noted by prior work [1], W must be in the uncovered set of the tournament graph, and one of the following two cases must be true.

- W defeats B.
- There exists a candidate $Y \in \mathcal{A}$ s.t. W defeats Y and Y defeats B.

For every $j \in [m-1]$, we now bound the probability of A_j being the winner. For every $j \in [m-1]$, we define Bernoulli random variables $\{Y_{i,j}\}_{i=1}^n$ denoting the event that voter i ranks candidate A_j over candidate B. From Equation 3, we have that $Y_{i,j} \sim \text{Bern}\left(g\left(\frac{d(i,A_j)}{d(i,B)}\right)\right)$. For every distinct $j, k \in [m-1]$, we define Bernoulli random variables $\{Z_{i,j,k}\}_{i=1}^n$ denoting the event that voter i ranks candidate A_j over A_k. $Z_{i,j,k} \sim \text{Bern}(g\left(\frac{d(i,A_k)}{d(i,A_j)}\right))$.

Observe that

$$\mathbb{P}[A_j \text{ wins}] \le \mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \ge \frac{n}{2} \bigcup_{k\in[m-1]\setminus\{j\}} \left(\sum_{i=1}^{n} Z_{i,j,k} \ge \frac{n}{2} \cap \sum_{i=1}^{n} Y_{i,k} \ge \frac{n}{2}\right)\right). \tag{48}$$

Let α_j denote the expected value of the random variable $\sum_{i=1}^{n} Y_{i,j}$, i.e., the expected number of voters who rank candidate A_j over B.

$$\alpha_j := \sum_{i=1}^{n} \mathbb{E}[Y_{i,j}] = \sum_{i=1}^{n} g\left(\frac{d(i,B)}{d(i,A_j)}\right) \text{ for every } j \in [m-1]. \tag{49}$$

Let $\beta_{j,k}$ denote the expected value of the random variable $\sum_{i=1}^{n} Z_{i,j,k}$, i.e., the expected number of voters who rank candidate A_j over A_k.

$$\beta_{j,k} := \sum_{i=1}^{n} \mathbb{E}[Z_{i,j,k}] = \sum_{i=1}^{n} g\left(\frac{d(i,A_k)}{d(i,A_j)}\right) \text{ for every } j \in [m-1]. \tag{50}$$

Similar to Equation (31), we have the following bound:

If $\alpha_j \le \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}$, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \ge \frac{n}{2}\right) = \mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \ge \alpha_j\left(1 + \frac{n}{2\alpha_j} - 1\right)\right) \tag{51}$$

$$\overset{(a)}{\le} \left(\frac{e^{(\frac{n}{2\alpha_j}-1)}}{(\frac{n}{2\alpha_j})^{n/2\alpha_j}}\right)^{\alpha_j} \tag{52}$$

$$\le \left(\frac{2\alpha_j}{n}\right)^2 \left(\frac{m\alpha_j}{n}\exp\left(-\frac{\alpha_j}{n/2-2}\right)\right)^{(\frac{n}{2}-2)} e^{\frac{n}{m}} \tag{53}$$

$$\overset{(c)}{\le} \left(\frac{2\alpha_j}{n}\right)^2 e^{\frac{n}{2}} \left(\left(1 - n^{-(\frac{1}{2}-\epsilon)}\right)\exp\left(-\frac{\frac{n}{2} - \frac{n^{(\frac{1}{2}+\epsilon)}}{2}}{n/2-2}\right)\right)^{\frac{n}{2}-2} \tag{54}$$

$$= \left(\frac{2\alpha_j}{n}\right)^2 \left(1 - n^{-(\frac{1}{2}-\epsilon)}\right)^{(n/2-2)} \exp\left(\frac{n^{(\frac{1}{2}+\epsilon)}}{2}\right) \tag{55}$$

$$\overset{(d)}{\le} \left(\frac{2\alpha_j}{n}\right)^2 \exp\left(\frac{-2n^{-(\frac{1}{2}-\epsilon)}(n/2-2)}{2 - n^{-(\frac{1}{2}-\epsilon)}} + \frac{n^{(\frac{1}{2}+\epsilon)}}{2}\right) \tag{56}$$

$$= \left(\frac{2\alpha_j}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{(2n^{(\frac{1}{2}-\epsilon)} - 1)2}\right) \tag{57}$$

From Equation (31) in the proof of Theorem 1, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_{i,j} \ge \frac{n}{2}\right) \le \left(\frac{2\alpha_j}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right) \text{ if } \alpha_j \le \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}. \tag{58}$$

$$\text{Similarly, } \mathbb{P}\left(\sum_{i=1}^{n} Z_{i,j,k} \ge \frac{n}{2}\right) \le \left(\frac{2\beta_{j,k}}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right) \text{ if } \beta_{j,k} \le \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}. \tag{59}$$

Consider two exhaustive cases on candidate A_j and define an event E_j for every $j \in [m-1]$. We compute the expected fraction of votes on pairwise comparisons. The event E_j denotes the existence of an at-most two hop directed path from a candidate A_j to candidate B for Copeland such that the expected fraction of votes on all edges along that path exceed $\frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}$. Recall that we only considered one hop path for the case of PLU in the proof of Theorem 1.

$$E_j := \left(\alpha_j \geq \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}\right) \bigcup_{k\in[m-1]\setminus\{j\}} \left(\left(\beta_{j,k} \geq \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}\right) \bigcap \left(\alpha_k \geq \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}\right)\right). \tag{60}$$

If E_j holds true, we can directly upper bound the ratio of the social cost of candidate A_j to the social cost of candidate B using Lemma 2, which in turn provides a bound on the distortion. If E_j does not hold, we apply the union bound and Chernoff's bound to upper bound the probability of A_j being the winner. By multiplying this probability bound with the ratio of social costs obtained from Lemma 2, we derive a bound on the distortion.

Define $S := \{j \in [m-1] : E_j \text{ is not true}\}$. Furthermore, we define $\mathcal{K}_1(j) := \{j \in [m-1] : \alpha_k \geq \beta_{j,k}\}$ and $\mathcal{K}_2(j) := \{j \in [m-1] : \alpha_k < \beta_{j,k}\}$ denotes complement of $\mathcal{K}_1(j)$ for every $j \in [m]$.

From Equations (58) and (59), both of the following conditions 1 and 2 are satisfied for every $j \in S$.

1. $\mathbb{P}\left(\sum_{i=1}^n Y_{i,j} \geq \frac{n}{2}\right) \leq \left(\frac{2\alpha_j}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right)$
2. For every $k \in [m-1] \setminus \{j\}$,

   $\mathbb{P}\left(\sum_{i=1}^n Z_{i,j,k} \geq \frac{n}{2}\right) \leq \left(\frac{2\beta_{j,k}}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right)$ if $k \in \mathcal{K}_1(j)$

   and, $\mathbb{P}\left(\sum_{i=1}^n Y_{i,k} \geq \frac{n}{2}\right) \leq \left(\frac{2\alpha_k}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right)$ if $k \in \mathcal{K}_2(j)$.

Furthermore, we define $\gamma_j := \max\left(\max_{k\in[m-1]\setminus\{j\}} \left(\min(\alpha_k, \beta_{j,k})\right), \alpha_j\right)$.

Since, for every Copeland winner W, it must either defeat B or there exists a $Y \in \mathcal{A}$ s.t. W defeats Y and Y defeats B. Using union bound for every $j \in S$, we have

$$\begin{aligned}
\mathbb{P}[A_j \text{ wins}] &\leq \mathbb{P}\left[\sum_{i=1}^n Y_{i,j} \geq \frac{n}{2}\right] + \sum_{k\in[m-1]\setminus\{j\}} \mathbb{P}\left[\left(\sum_{i=1}^n Y_{i,k} \geq \frac{n}{2}\right) \cap \left(\sum_{i=1}^n Z_{i,j,k} \geq \frac{n}{2}\right)\right] \text{ if } j \in S \\
&\leq \left(\frac{2\alpha_j}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right) + \sum_{k\in\mathcal{K}_2(j)} \left(\frac{2\alpha_k}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right) \\
&\qquad + \sum_{k\in\mathcal{K}_1(j)} \left(\frac{2\beta_{j,k}}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right) \text{ if } j \in S \\
&\leq m\left(\frac{2\gamma_j}{n}\right)^2 \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right) \text{ if } j \in S.
\end{aligned} \tag{61}$$

The last inequality follows from the definition of γ_j.

Furthermore from Lemma 2 and the definition of γ_j, [7] we have

$$\frac{\text{SC}(A_j,d)}{\text{SC}(B,d)} \leq \left(\max\left(\frac{n}{\gamma_j}\hat{g}_{\text{MID}} - 1, \frac{n}{\gamma_j}\hat{g}_{\text{OUT}} + 1\right)\right)^2 \tag{62}$$

Using Equation (62) and (61) and applying $\max(a,b) \leq a+b$, we have

$$\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j,d)}{\text{SC}(B,d)} \leq 4m\exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)}+8}{2(2n^{(\frac{1}{2}-\epsilon)}-1)}\right)(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}})^2 \text{ if } j \in S. \tag{63}$$

[7]This follows on splitting $\frac{SC(A_j,d)}{SC(B,d)} = \frac{SC(A_j,d)}{SC(A_k,d)} \times \frac{SC(A_k,d)}{SC(B,d)}$ and applying the lemma separately. We further use the fact that $\frac{1}{\gamma} = \min\left(\min_{k\in[m-1]\setminus\{j\}}\left(\max(\frac{1}{\alpha_k}, \frac{1}{\beta_{j,k}})\right), \frac{1}{\alpha_j}\right)$

Recall that for every $j \in [m-1] \setminus S$, E_j is satisfied. Let us further denote

$$\hat{E}_j := \alpha_j \geq \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2} \text{ and } \hat{D}_{j,k} := \left(\beta_{j,k} \geq \frac{n}{2} - \frac{n^{(1/2+\epsilon)}}{2}\right).$$

Observe that E_j being satisfied implies either a) $\hat{E}_j$ is satisfied or b) $\exists k \in [m-1] \setminus \{j\}$ s.t $\hat{E}_k$ and $\hat{D}_{j,k}$ are satisfied. We consider both cases separately.

Suppose $\hat{E}_j$ is satisfied for some $j \in [m-1] \setminus S$. Then we have from Lemma 2,

$$\frac{SC(A_j, d)}{SC(B, d)} \leq \max\left(\frac{2\hat{g}_{\text{MID}}}{(1 - n^{-(1/2-\epsilon)})} - 1, \frac{2\hat{g}_{\text{OUT}}}{(1 - n^{-(1/2-\epsilon)})} + 1\right). \tag{64}$$

Now we consider case (b) where $\hat{E}_k$ and $\hat{D}_{j,k}$ are both satisfied for some $k \in [m-1] \setminus \{j\}$. From Lemma 2 we have,

$$\frac{SC(A_j, d)}{SC(B, d)} \leq \max\left(\left(\frac{2\hat{g}_{\text{MID}}}{(1 - n^{-(1/2-\epsilon)})} - 1\right)^2, \left(\frac{2\hat{g}_{\text{OUT}}}{(1 - n^{-(1/2-\epsilon)})} + 1\right)^2\right). \tag{65}$$

Now combining Equations (63), (64), and (65), we have for any metric space $d \in \mathcal{M}(\mathcal{N} \cup \mathcal{A})$,

$$\begin{aligned}
\text{DIST}^{(g)}(\text{COP}, n, m) &\leq \left(\sum_{j \in S}\left(\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j, d)}{\text{SC}(B, d)}\right) + \mathbb{P}[B \text{ wins}] + \sum_{j \in [m-1] \setminus S}\left(\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j, d)}{\text{SC}(B, d)}\right)\right) \\
&\overset{(a)}{\leq} 4(m-1)m \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right)(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}}) + \max\left(\max_{j \in [m-1] \setminus S} \frac{SC(A_j, d)}{SC(B, d)}, 1\right) \\
&\overset{(b)}{\leq} 4(m-1)m \exp\left(\frac{-n^{(\frac{1}{2}+\epsilon)} + 8}{2(2n^{(\frac{1}{2}-\epsilon)} - 1)}\right)(\hat{g}_{\text{MID}} + \hat{g}_{\text{OUT}}) + \max\left(\left(\frac{2\hat{g}_{\text{MID}}}{(1 - n^{-(1/2-\epsilon)})} - 1\right)^2, \left(\frac{2\hat{g}_{\text{OUT}}}{(1 - n^{-(1/2-\epsilon)})} + 1\right)^2\right).
\end{aligned}$$

(a) follows from Equation (61) and the fact that $\sum_{j \in S}\left(\mathbb{P}[A_j \text{ wins}]\frac{\text{SC}(A_j,d)}{\text{SC}(B,d)}\right) + \mathbb{P}[B \text{ wins}] \leq \max\left(\max_{j \in S} \frac{SC(A_j,d)}{SC(B,d)}, 1\right)$.

(b) follows from combining Equations (63), (64), and (65). ☐

F Proof of Theorem 4

Theorem (Restatement of Theorem 4). $\text{DIST}^{(g)}(\text{RD}, m, n) \leq (m-1)\hat{g}_{\text{MID}} + 1$.

Proof. The probability of voter i voting for candidate W as its top candidate is upper bounded by $g\left(\frac{d(i,B)}{d(i,W)}\right)$ which is the probability that W is ranked over B. Therefore, under RD, the probability of W winning satisfies:

$$\mathbb{P}[W \text{ wins}] \leq \frac{1}{n}\left(\sum_{i=1}^{n} g\left(\frac{d(i, B)}{d(i, W)}\right)\right). \tag{66}$$

Recall that we define the set of candidates in $\mathcal{A} \setminus B$ as $\{A_1, A_2, \ldots, A_{m-1}\}$. In the rest of the analysis we denote $d(i, A_j)$ by $y_{i,j}$ (for all $j \in [m-1]$) and $d(i, B)$ by b_i for every $i \in [n]$. We also denote $d(B, A_j)$ by z_j for every

$j \in [m-1]$. Now for every metric d, we bound the distortion as follows.

$$\textsc{dist}^{(g)}(\text{RD}, m, n) \leq \sum_{j=1}^{m-1} \left(\mathbb{P}[A_j \text{ wins}] \frac{\sum_{i=1}^{n} y_{i,j}}{\sum_{i=1}^{n} b_i} \right) + (1 - \sum_{j=1}^{m-1} \mathbb{P}[A_j \text{ wins}]) \tag{67}$$

$$= \sum_{j=1}^{m-1} \mathbb{P}[A_j \text{ wins}] \left(\frac{\sum_{i=1}^{n} y_{i,j}}{\sum_{i=1}^{n} b_i} - 1 \right) + 1 \tag{68}$$

$$\overset{(a)}{\leq} \sum_{j=1}^{m-1} \frac{1}{n} \left(\sum_{i=1}^{n} g\left(\frac{b_i}{y_{i,j}} \right) \right) \frac{\sum_{i=1}^{n} (y_{i,j} - b_i)}{\sum_{i=1}^{n} b_i} + 1 \tag{69}$$

$$\leq \sum_{j=1}^{m-1} \frac{1}{n} \left(\sum_{i=1}^{n} g\left(\frac{b_i/z_j}{y_{i,j}/z_j} \right) \right) \frac{\sum_{i=1}^{n} (y_{i,j}/z_j - b_i/z_j)}{\sum_{i=1}^{n} b_i/z_j} + 1 \tag{70}$$

$$\overset{(d)}{\leq} \sum_{j=1}^{m-1} \frac{\left(\sum_{i=1}^{n} g\left(\frac{b_i/z_j}{y_{i,j}/z_j} \right) \right)}{\sum_{i=1}^{n} b_i/z_j} + 1 \tag{71}$$

$$\overset{(e)}{\leq} (m-1) \frac{g\left(\frac{x^*_{\text{MID}}}{1 - x^*_{\text{MID}}} \right)}{x^*_{\text{MID}}} + 1 = (m-1)\hat{g}_{\text{MID}} + 1. \tag{72}$$

(a) follows from Equation (66).

(d) follows from the fact that $y_{i,j} - b_i \leq z_j$ which follows from triangle inequality.

(e) follows from the following arguments by considering two cases namely $\frac{b_i}{z_j} \leq 1$ and $\frac{b_i}{z_j} \geq 1$.

When $\frac{b_i}{z_j} \leq 1$ and thus, $\frac{y_{i,j}}{z_j} \geq 1 - \frac{b_i}{z_j}$ from triangle inequality. Similarly, we have $\frac{y_{i,j}}{z_j} \geq \frac{b_i}{z_j} - 1$ when $\frac{b_i}{z_j} \geq 1$. Thus,

$$\frac{g\left(\frac{b_i/z_j}{y_{i,j}/z_j} \right)}{b_i/z_j} \leq \max \left(\sup_{x \in (0,1)} \frac{g(\frac{x}{1-x})}{x}, \sup_{x \in (1,\infty)} \frac{g(\frac{x}{x-1})}{x} \right) \text{ for every } i \in [n] \tag{73}$$

$$\implies \frac{\sum_{i=1}^{n} g\left(\frac{b_i/z_j}{y_{i,j}/z_j} \right)}{\sum_{i=1}^{n} b_i/z_j} \leq \max \left(\frac{g\left(\frac{x^*_{\text{MID}}}{1 - x^*_{\text{MID}}} \right)}{x^*_{\text{MID}}}, 1 \right). \tag{74}$$

The last inequality follows from the fact that $\frac{g(\frac{x}{x-1})}{x} \leq 1$ when $x \geq 1$. Further, we have $\hat{g}_{\text{MID}} \geq 1$ for all valid g. ☐

G Proof of Theorem 6

Theorem (Restatement of Theorem 6). Let $\textsc{dist}^{\theta}_{PL}(\text{RD}, m, n)$ denote the distortion when the voters' rankings are generated per the PL model with parameter θ. We have $\lim_{n \to \infty} \textsc{dist}^{\theta}_{PL}(\text{RD}, m, n) \geq 1 + \frac{(m-1)^{1/\theta}}{2}$.

Proof. We have a 1-D Euclidean construction. Let B be at 0 and all other candidates $\mathcal{A} \setminus \{B\}$ be at 1. $m-1$ voters are at 0, and one voter is at t. We will set t later by optimizing for the distortion.

The distortion for this instance is $\mathbb{P}[B \text{ wins}] \cdot 1 + \mathbb{P}[B \text{ loses}] \cdot \frac{n-t}{t} = \frac{n-1}{n} + \frac{1}{n} \frac{t^{-\theta}}{t^{-\theta} + (m-1)(1-t)^{-\theta}} + \frac{1}{n} \frac{(m-1)(1-t)^{-\theta}}{t^{-\theta} + (m-1)(1-t)^{-\theta}} \frac{n-t}{t}$. We drop the terms which are $O(1/n)$ to obtain $1 + \frac{(m-1)(1-t)^{-\theta}}{t(t^{-\theta} + (m-1)(1-t)^{-\theta})}$. This simplifies to $1 + \frac{(m-1)t^{\theta-1}}{(1-t)^{\theta} + (m-1)t^{\theta}}$. This is lower bounded by $1 + \frac{(m-1)t^{\theta-1}}{1 + (m-1)t^{\theta}}$. Setting $t = (m-1)^{-1/\theta}$, we obtain a distortion lower bound of $1 + \frac{(m-1)^{1/\theta}}{2}$. ☐