Proportional Representation in Rank Aggregation

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In rank aggregation, the task is to aggregate multiple weighted input rankings into a single output ranking. While numerous methods, so-called social welfare functions (SWFs), have been suggested for this problem, all of the classical SWFs tend to be majoritarian and are thus not acceptable when a proportional ranking is required. Motivated by this observation, we will design SWFs that guarantee that every input ranking is proportionally represented by the output ranking. Specifically, our central fairness condition requires that the number of pairwise comparisons between candidates on which an input ranking and the output ranking agree is proportional to the weight of the input ranking. As our main contribution, we present a simple SWF called the Proportional Sequential Borda rule, which satisfies this condition. Moreover, we introduce two variants of this rule: the Ranked Method of Equal Shares, which has a more utilitarian flavor while still satisfying our fairness condition, and the Flow-adjusting Borda rule, which satisfies an even stronger fairness condition. Many of our axioms and techniques are inspired by results on approval-based committee voting and participatory budgeting, where the concept of proportional representation has been studied in depth.

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1 Introduction

For booking flights or hotels, many users consult aggregator websites such as google flights, skyscanner, or booking.com. These websites allow users to quickly get an overview of the available flights or hotels by offering several ways to sort the options. For instance, flights may be sorted by their price, travel time, or number of stop-overs, whereas hotels are commonly sorted by price or user rating. Furthermore, aggregator websites typically offer a recommended ranking that combines multiple criteria.[1] More generally, it would be desirable to allow users to specify importance weights for individual criteria and to compute customized output rankings based on these weights.

The problem of finding such aggregate rankings is commonly studied under the term *rank aggregation* and has attracted significant attention in social choice theory [e.g., 2, 7] and beyond [e.g., 15, 25, 43, 47]. In more detail, in rank aggregation, we get a profile of weighted rankings as input, where the weights are non-negative and add up to one, and we need to return a single output ranking. For instance, this model captures situations where users want to sort hotels to 60% by their price and 40% by their user rating as this task requires us to combine the ranking by price and the ranking by user rating. Moreover, in social choice theory a multitude of rank aggregation methods, so-called *social welfare functions (SWFs)*, have been suggested, with the most prominent examples including the Kemeny rule [22, 51], various types of scoring rules [6, 16, 30, 46], and Condorcet-type rules [17, 19].

However, as observed by Lederer et al. [31], none of these classic SWFs is suitable for aggregating rankings based on user-specified weights because they are heavily majoritarian. For example, if a user wants to combine two inverse rankings with weights of 51% and 49%, most SWFs will simply return the ranking with the larger weight instead of actually combining the rankings. Motivated by this observation, Lederer et al. [31] have initiated the study of proportional SWFs, aiming for methods that represent the input rankings proportional to their weights.[2] Specifically, these authors formalize proportional representation in terms of the number of pairs of candidates on which the input and output rankings agree on: an input ranking with weight α should agree at least with an α fraction of all pairwise comparisons of the output ranking. Moreover, Lederer et al. [31] suggest the Squared Kemeny rule to compute proportional rankings. However, while the Squared Kemeny rule is certainly more proportional than established SWFs, it does not satisfy the aforementioned fairness condition in general, thus leaving the design of fully proportional SWFs as an open problem.

1.1 Our Contribution

In this paper, we will design the first truly proportional SWFs by employing ideas from approval-based committee voting and participatory budgeting, two fields in which proportionality has been studied extensively [see, e.g., 29, 41]. To explain our results, we define the *utility* of an input ranking $\succ$ for an output ranking $\rhd$ as the number of pairwise comparisons these rankings agree on, i.e., $u(\succ, \rhd) = |\{(x, y) \colon x \succ y \text{ and } x \rhd y\}|$. Now, since every ranking on m candidates induces $\binom{m}{2}$ pairwise comparisons between candidates, the fairness axiom of Lederer et al. [31] formally requires that every input ranking with weight α obtains a utility of at least $\lfloor \alpha \binom{m}{2} \rfloor$ from the output ranking. We call this condition *unanimous proportional justified representation (uPJR)* because, when viewing rankings as approval ballots over pairs of candidates, this axiom weakens a well-known proportionality notion called proportional justified representation [42] by only focusing on

[1] See, e.g., https://www.skyscanner.net/media/how-skyscanner-works for an overview of how skyscanner computes its recommended ranking.

[2] We note that proportionality is often interpreted as a fairness notion for the input rankings. In rank aggregation, there is another influential line of work that investigates fairness with respect to attributes of candidates [e.g., 12, 38, 48]. Specifically, the idea of these papers is that some candidates have protected attributes that should be fairly represented in the output ranking, regardless of the information provided of the input rankings.

groups of voters that unanimously report the same ballot. Based on this relation to approval-based committee voting, we also introduce two further fairness notions for rank aggregation called *unanimous justified representation (uJR)* and *strong proportional justified representation (sPJR)*, which respectively weaken and strengthen uPJR. As our main contribution, we propose SWFs that satisfy these fairness conditions. Specifically, we will show the following results:

- As a warm-up, we will first analyze uJR, which demands that every input ranking with a weight of at least $1/\binom{m}{2}$ obtains a non-zero utility from the output ranking. We note that uJR can be seen as the counterpart of justified representation, another well-known proportionality notion in approval-based committee voting [3]. We show that the Squared Kemeny rule by Lederer et al. [31] severely fails this condition (Proposition 1), thus demonstrating the need of more proportional rules. Moreover, we also design a simple rule inspired by Chamberlin-Courant approval voting that satisfies uJR (Proposition 2).
- We next turn to uPJR and, inspired by similar results in approval-based committee voting and participatory budgeting [8, 37], prove that uPJR is implied by a more structured fairness axiom called rank-priceability (Proposition 3). Based on this insight, we design a simple SWF, the *Proportional Sequential Borda rule (*PSB*)*, that satisfies rank-priceability and thus uPJR (Theorem 1). Roughly, PSB repeatedly picks the Borda winner in the current profile, deletes this candidate from the profile, and reduces the weights of each input ranking proportional to its contribution to the score of the Borda winner.
- Thirdly, we introduce the *Ranked Method of Equal Shares (*RMES*)* as a more utilitarian variant of PSB. Moreover, this SWF closely resembles the Method of Equal Shares, one of the most prominent tools for proportional decision making [36, 37]. Roughly, RMES distributes a budget of $\binom{m}{2}$ to the input rankings proportional to their weights, which is used to repeatedly buy candidates. In particular, in every step, the affordable candidate that minimizes the cost per utility ratio for the input rankings is bought. While this method satisfies rank-priceability and thus uPJR (Theorem 3), it is also rather utilitarian: the first $\lfloor \frac{m}{4} \rfloor$ candidates in this ranking (or roughly $\frac{7}{16}$ of the total utility) agree with the majoritarian ranking obtained by repeatedly picking the Borda winner and deleting it from the profile (Proposition 4).
- We further analyze sPJR which extends the reasoning of PJR to arbitrary groups of input rankings: if a group of input rankings has a total weight of α, the output ranking should agree with at least $\lfloor \alpha \binom{m}{2} \rfloor$ pairwise comparisons of these rankings. As we will show, both PSB and RMES fail this condition because, roughly, rank-priceability does not entail guarantees for groups of rankings. We thus introduce a refinement of rank-priceability called pair-priceability and show that this notion implies sPJR (Proposition 5). Moreover, we propose another variant of PSB called the Flow-adjusting Borda rule (FB) that satisfies rank-priceability (Theorem 5). Notably, FB only augments PSB by using a more sophisticated scheme for updating the weights of the input rankings.
- Inspired by an analogous result for the Squared Kemeny rule [31, Theorem 4.2], we examine the average utility our SWFs guarantee to a group of rankings, as a function of the total weight of the group. For each of our SWFs, we show that the average utility of every group of rankings is at least linear in the weight of the group. In more detail, we prove that the average utility of every group of rankings with a total weight of α is at least $\frac{\alpha}{4}\binom{m}{2} - \frac{3}{16}$, thus providing another strong proportionality guarantee for our rules (Theorems 2, 4 and 6).

1.2 Related Work

Rank aggregation is one of the oldest problems in social choice theory as even Arrow's impossibility theorem has originally been shown in this setting [1]. There is thus a large body of work on SWFs and we refer to the textbooks by Arrow et al. [2] and Brandt et al. [7] for an overview of this research area.

Our paper is related to an influential stream of works that studies scoring rules for rank aggregation [e.g,. 6, 20, 22, 46, 50, 51]. Specifically, positional scoring rules such as the Borda rule compute the output ranking by assigning scores to the candidates and sorting the candidates in descending order of their scores. Such SWFs have attracted significant attention and have, e.g., been repeatedly characterized based on a population-consistency condition [e.g., 34, 46, 49]. Moreover, positional scoring rules can be easily modified to work sequentially by repeatedly adding the candidate with the highest (resp. lowest) score to the next best (resp. worst) position of the output ranking [e.g., 6, 20, 46]. The Proportional Sequential Borda rule and the Flow-adjusting Borda rule are closely related to this approach as we repeatedly add the Borda winner to the ranking, but we carefully update the weights of the input rankings to obtain a proportional outcome. Furthermore, our work is related to the Kemeny rule, another well-studied SWF [e.g., 11, 22, 50, 51], as this rule maximizes the utilitarian social welfare in our setting.

While the aforementioned works are influential, they do not focus on proportional decision making. Indeed, the study of proportionality in rank aggregation has only recently been initiated by Lederer et al. [31] who suggested the Squared Kemeny rule to compute proportional rankings. Moreover, Aziz et al. [4] have studied a prefix-based fairness concept for rank aggregation, which is unrelated to uPJR. We will thus rely on ideas from approval-based committee voting [29] and participatory budgeting [5, 41] to formalize proportionality. In particular, all our fairness axioms are closely related to fairness notions from this literature [e.g., 3, 8, 37, 42]. Moreover, the budgeting approach used for defining our SWFs is reminiscent of the Method of Equal Shares, one of the most prominent rules in these settings [36, 37].

Inspired by these works on approval-based committee voting and participatory budgeting, proportionality has been studied in numerous other models, some of which are related to rank aggregation. In particular, Skowron et al. [45] and Brill and Israel [9] study the problem of proportionally aggregating the voters' approval ballots into a ranking, but they focus on a prefix-based fairness concept. Moreover, there are models of repeated decision making [10, 14, 26, 27], where a candidate needs to be selected in each round and each group of voters should be fairly represented over all rounds. However, as the sequence of candidates is not interpreted as a ranking, this literature again focuses on different fairness notions. Finally, Masařík et al. [33] study a general model of proportional decision making, which contains rank aggregation as a special case. However, applying these results to rank aggregation gives only very mild guarantees and, e.g., allow that groups of agents of size less than $\frac{1}{m}$ are left without any representation.

Finally, rank aggregation has also gained significant attention outside of social choice theory. For example, the problem of rank aggregation is considered in computational biology [e.g., 25, 32], machine learning [39, 43, 47], metasearch [18, 40], and crowdsourcing [15, 35]. We thus believe that our work has also the potential to provide novel insights for these applications of rank aggregation.

2 Preliminaries

Let $C = \{x_1, \dots, x_m\}$ denote a set of m candidates. A *ranking* $\succ$ is a strict linear order over C, and we typically write rankings as comma-separated lists. For instance, $\succ = x_1, x_2, x_3$ means that x_1 is preferred to x_2 and x_2 is preferred to x_3. The set of all rankings over C will be denoted by $\mathcal{R}$. Following Lederer et al. [31], we define a *ranking profile* R as a function from $\mathcal{R}$ to $[0, 1]$ such that $\sum_{\succ \in \mathcal{R}} R(\succ) = 1$. Less formally, a ranking profile specifies for every ranking $\succ \in \mathcal{R}$ a weight $R(\succ)$ and the total weight sums up to one. These weights may be interpreted as the importance scores in a multi-criteria decision-making problem or as the shares of voters that report a given ranking. Furthermore, we say a function $S : \mathcal{R} \to [0, 1]$ is a *subprofile* of a profile R if $S(\succ) \leq R(\succ)$ for all

$\succ \in \mathcal{R}$. In a voting setting, a subprofile S can be interpreted as an arbitrary group of voters. We define the weight of a subprofile S by $|S| = \sum_{\succ \in \mathcal{R}} S(\succ)$ and observe that $|S| \in [0, 1]$.

Given a ranking profile R, our goal is to aggregate the input rankings into one output ranking. For this problem, we use *social welfare functions (SWFs)*, which are functions that map every ranking profile to a single output ranking. To clearly distinguish between input rankings and output rankings, we will write $\succ$ for the former and $\rhd$ for the latter. The assumption that SWFs always choose a single output ranking will sometimes require tie-breaking as multiple rankings can be tied for the win. We will typically break such ties in favor of candidates with smaller indices and note that this assumption does not affect our results. Indeed, all our results also hold when viewing our rules as set-valued SWFs, but this model introduces unnecessary notational complexity.

2.1 Proportionality Axioms

The central goal of this paper is to find output rankings that represent the input rankings proportionally to their weights: a ranking with weight α should have an influence of α on the outcome. Following the approach of Lederer et al. [31], we will formalize this idea by requiring that the number of pairs of candidates for which an input ranking and the output ranking agree is proportional to the weight of the input ranking. We moreover note that we define our fairness axioms as properties of rankings; an SWF f satisfies a given axiom if its chosen ranking $f(R)$ satisfies the axiom for all profiles R.

To formalize our fairness axioms, we define the *utility* of a ranking $\succ$ for another ranking $\rhd$ by $u(\succ, \rhd) = |\{(x, y) \in C^2 : x \succ y \land x \rhd y\}|$. That is, the utility of an input ranking $\succ$ for the output ranking $\rhd$ is the number of pairs of candidates for which the rankings agree. Furthermore, we let $u(\succ, x, X) = |\{y \in X \setminus \{x\} : x \succ y\}|$ denote the utility of a candidate x within the set of candidates X with respect to $\succ$. Alternatively, $u(\succ, x, X)$ can also be interpreted as the Borda score of x within the set X. This term will be crucial in our analysis because $u(\succ, \rhd) = \sum_{i=1}^{m-1} u(\succ, x_i, \{x_i, \dots, x_m\})$ for every input ranking $\succ$ and output ranking $\rhd = x_1, \dots, x_m$. We note that the utility $u(\succ, \rhd)$ is dual to the swap distance $swap(\succ, \rhd) = |\{(x, y) \in C^2 : x \succ y \land y \rhd x\}|$ used by Lederer et al. [31]. Specifically, it holds for all rankings $\succ, \rhd \in \mathcal{R}$ that $u(\succ, \rhd) = \binom{m}{2} - swap(\succ, \rhd)$ since $\binom{m}{2}$ is the maximal utility (or swap distance) for two rankings with m candidates. Therefore, our results could also be phrased in terms of swap distance.

We will now introduce our first fairness condition called unanimous proportional justified representation (uPJR), which requires that the utility of every ranking should be proportional to its weight. Both Lederer et al. [31] and Aziz et al. [4, Section 8] have investigated this condition but only present SWFs that approximate uPJR.

Definition 1 (Unanimous Proportional Justified Representation)**.** A ranking $\rhd$ satisfies *unanimous proportional justified representation (uPJR)* for a profile R if $u(\succ, \rhd) \geq \lfloor R(\succ) \cdot \binom{m}{2} \rfloor$ for all $\succ \in \mathcal{R}$.

The name of this axiom is motivated by the fact that uPJR can be seen as a weakening of proportional justified representation (PJR), a well-known fairness condition for approval-based committee voting [42]. In more detail, in approval-based committee voting, a set of voters $N = \{1, \dots, n\}$ report approval ballots $A_i \subseteq C$ over the candidates and the goal is to choose a subset of the candidates of predefined size k. Then, the idea of PJR is that, if a group of voters S is large enough to deserve ℓ seats and the voters in S agree on ℓ approved candidates, the winning committee should contain at least ℓ candidates that are approved by voters in S. More formally, a committee W satisfies PJR for an approval profile A if $|W \cap \bigcup_{i \in S} A_i| \geq \ell$ for every group of voters S with $|S| \geq \frac{\ell n}{k}$ and $|\bigcap_{i \in S} A_i| \geq \ell$.

PJR can be naturally adapted to rank aggregation by associating each ranking $\succ$ with the set $A(\succ) = \{(x_i, x_j) \in C \times C : x_i \succ x_j\}$, which can be interpreted as an approval ballot over

$C^2 = \{(x_i, x_j) \in C \times C \colon x_i \neq x_j\}$. Hence, we can view the problem of rank aggregation as an instance of approval-based committee voting over the set C^2 with transitivity constraints. Specifically, given the input ballots $A(\succ)$ with weights $R(\succ)$, we need to choose a transitive subset of C^2 of size $k = \binom{m}{2}$. Applying PJR to this instance of committee voting results in the following condition: a ranking $\rhd$ satisfies PJR for a profile R if $|A(\rhd) \cap \bigcup_{\succ \in \mathcal{R} \colon S(\succ) > 0} A(\succ)| \geq \ell$ for every subprofile S of R with $|S| \geq \ell / \binom{m}{2}$ and $|\bigcap_{\succ \in \mathcal{R} \colon S(\succ) > 0} A(\succ)| \geq \ell$. Finally, uPJR arises from PJR by additionally requiring that S only assigns positive weight to a single ranking.

In addition to uPJR, we will consider two more fairness axioms in this paper. The first one, unanimous justified representation, is a weakening of uPJR which requires that each rankings with a weight of at least $1/\binom{m}{2}$ should get a non-zero utility. We observe that this axiom can be seen as the counterpart to justified representation (JR), another well-known fairness condition in approval-based committee elections [3].[3]

Definition 2 (Unanimous Justified Representation (uJR))**.** A ranking $\rhd$ satisfies *unanimous justified representation (uJR)* for a profile R if $u(\succ, \rhd) \geq 1$ for every $\succ \in \mathcal{R}$ with $R(\succ) \geq 1/\binom{m}{2}$.

Secondly, we will also consider a strengthening of PJR, which extends the reasoning of uPJR to arbitrary groups of rankings. Specifically, sPJR requires that for every subprofile of weight α, the output ranking chooses α pairwise comparisons from the union of the rankings in S. We note that sPJR does not impose any cohesiveness conditions, so it is a more demanding proportionality axiom than PJR and uPJR.

Definition 3 (Strong Proportional Justified Representation)**.** A ranking $\rhd$ satisfies *strong proportional justified representation (sPJR)* if $|A(\rhd) \cap \bigcup_{\succ \in \mathcal{R} \colon S(\succ) > 0} A(\succ)| \geq \lfloor |S| \cdot \binom{m}{2} \rfloor$ for all subprofiles S of R.

Finally, we will also quantitatively measure the fairness of SWFs. Specifically, following Lederer et al. [31] and Skowron and Górecki [44], we will derive lower bounds on the average utility of an arbitrary subprofile S, as a function of the size of $|S|$. This approach allows for a much more fine-grained analysis than an axiomatic analysis. On the down side, the assumption that S can be an arbitrary subprofile restricts the guarantees we can show. For instance, when $|S| = 1$, the best bound one can prove is $\frac{1}{2}\binom{m}{2}$, which is the average utility of every output ranking for the profile where two inverse rankings each have a weight of $\frac{1}{2}$.

3 Results on Unanimous Justified Representation

As a warm-up, we will start by examining uJR. We first note that traditional SWFs, such as the Kemeny rule or the Borda rule, fail this condition because these rules are heavily majoritarian. We will thus focus on the Squared Kemeny rule which has been explicitly proposed by Lederer et al. [31] for computing proportional rankings. In more detail, we will show that this SWF fails uJR arbitrarily badly as m increases (Section 3.1). Furthermore, in Section 3.2, we will devise a simple rule inspired by Chamberlin-Courant approval voting that satisfies uJR.

3.1 The Squared Kemeny Rule

We start by analyzing the Squared Kemeny rule, which chooses the ranking that minimizes the total squared swap distance to the input rankings. Formally, the *Squared Kemeny rule (*SqK*)* is defined by

[3] uJR is equivalent to JR in rank aggregation. In particular, the latter axiom requires that $|A(\rhd) \cap \bigcup_{\succ \in \mathcal{R} \colon S(\succ) > 0} A(\succ)| \geq 1$ if $|S| \geq \binom{m}{2}^{-1}$ and $|\bigcap_{\succ \in \mathcal{R} \colon S(\succ)} A(\succ)| \geq 1$ for all subprofiles S. Now, if S assigns positive weight to two rankings $\succ_1$ and $\succ_2$, there is a pair of candidates x_1, x_2 such that $x_1 \succ_1 x_2$ and $x_2 \succ_2 x_1$. Consequently, every output ranking agrees with either $\succ_1$ or $\succ_2$ in at least one pair of candidates. Hence, JR is trivial in rank aggregation unless S assigns a positive weight to a single ranking and it reduces to uJR in this case.

$\mathsf{SqK}(R) = \arg\min_{\rhd\in\mathcal{R}} \sum_{\succ\in\mathcal{R}} R(\succ)\cdot swap(\succ,\rhd)^2$ or, equivalently, $\mathsf{SqK}(R) = \arg\min_{\rhd\in\mathcal{R}} \sum_{\succ\in\mathcal{R}} R(\succ)\cdot (\binom{m}{2} - u(\succ,\rhd))^2$. We note that there can be multiple rankings that minimize the squared swap distance, so a full definition of this rule requires further tie-breaking. However, the tie-breaking will not matter for our subsequent proposition, so we omit these details.

We will next prove that SqK fails uJR for all $m \geq 5$. Specifically, we will present a family of profiles R such that $R(\succ) = \frac{m}{5}/\binom{m}{2}$ for some ranking $\succ$ but SqK uniquely chooses the inverse ranking of $\succ$. This means that the Squared Kemeny rule does not even approximate uJR: for every $k \in \mathbb{N}$, there is a number of candidates m, a profile R, and a ranking $\succ$ such that $\succ$ deserves a utility of k in R but obtains a utility of 0.

Proposition 1. *For all $m \geq 5$, there is a profile R and ranking $\succ$ such that $R(\succ) = \frac{m}{5}/\binom{m}{2}$ and $u(\succ, \mathsf{SqK}(R)) = 0$.*

Proof Sketch. We consider four rankings to prove this proposition: $\succ_1$ is given by $\succ_1 = x_1, x_2, \dots, x_m$, $\succ_2$ by $\succ_2 = x_1, x_m, \dots, x_2$, $\succ_3$ is an arbitrary ranking that bottom-ranks x_1 and agrees with $\lfloor \frac{1}{2}\binom{m-1}{2} \rfloor$ pairwise comparisons with $\succ_1$, and $\succ_4$ also bottom-ranks x_1 and arranges the candidates $x_2, \dots, x_m$ inversely to $\succ_3$. For example, if $m = 5$, we may choose the following rankings.

$$\succ_1 = x_1, x_2, x_3, x_4, x_5 \qquad \succ_2 = x_1, x_5, x_4, x_3, x_2$$
$$\succ_3 = x_5, x_2, x_3, x_4, x_1 \qquad \succ_4 = x_4, x_3, x_2, x_5, x_1$$

Further, let R denote the profile given by $R(\succ_1) = \frac{m}{5}\cdot\binom{m}{2}^{-1}$ and $R(\succ_2) = R(\succ_3) = R(\succ_4) = \frac{1}{3}\cdot(1 - \frac{m}{5}\cdot\binom{m}{2}^{-1})$. We show that the Squared Kemeny rule picks the ranking $\mathsf{SqK}(R) = x_m, \dots, x_1$ for R, thus leaving $\succ_1$ without representation. While the proof of this claim is tedious, we note three high-level ideas. First, it can be shown that the output ranking must generate a higher utility for $\succ_2$ than for $\succ_1$ because $R(\succ_2) > R(\succ_1)$. Secondly, we show that the closer the output ranking $\rhd$ without x_1 is to $x_m, \dots, x_2$, the lower we must rank x_1 in $\rhd$. For instance, if $x_5 \rhd x_4 \rhd \dots \rhd x_2$, we get that x_1 must be bottom-ranked. Thirdly, we prove that if x_1 is ranked sufficiently low in $\rhd$, then it is optimal to order the candidates $x_5, \dots, x_2$ inverse to $\succ_1$ because the weight of $\succ_2$ is significantly larger than that of $\succ_1$. By formalizing these ideas, it follows that $\mathsf{SqK}(R) = x_m, \dots, x_1$, thus proving the proposition. ◻

3.2 The Chamberlin-Courant SWF

In light of Proposition 1, one may think that involved techniques are required to design SWFs satisfying uJR. We will next refute this hypothesis by introducing a very simple SWF inspired by Chamberlin-Courant approval voting that satisfies this fairness condition. To this end, we recall that Chamberlin-Courant approval voting is an approval-based committee voting rule which chooses the committee that maximizes the number of voters who approve at least one selected candidate [13, 28]. Put differently, this rule maximizes the number of voters that have a utility of at least 1. We adapt this idea to the context of rank aggregation by defining the score function $s : \mathbb{N}_0 \to \mathbb{R}$ given by $s(x) = 1$ if $x > 0$ and $s(0) = 0$. Then, the *Chamberlin-Courant SWF* chooses a ranking $\rhd$ that maximizes $\sum_{\succ\in\mathcal{R}} R(\succ)\cdot s(u(\succ,\rhd))$, with ties broken arbitrarily. We will next show that this SWF satisfies uJR.

Proposition 2. *The Chamberlin-Courant SWF satisfies uJR.*

Proof. Fix a profile R and let $\rhd$ denote the ranking chosen by the Chamberlin-Courant SWF. Hence, $\rhd$ maximizes $\sum_{\succ\in\mathcal{R}} R(\succ)\cdot s(u(\succ,\rhd))$ and therefore also $\sum_{\succ\in\mathcal{R}} R(\succ)\cdot (s(u(\succ,\rhd)) - 1)$. We next observe that it holds for all $\succ, \rhd' \in \mathcal{R}$ that $s(u(\succ,\rhd')) = 0$ if and only if $\succ$ orders the candidates inversely to $\rhd'$. This means that $\sum_{\succ\in\mathcal{R}} R(\succ)\cdot (s(u(\succ,\rhd')) - 1) = -R(\blacktriangleleft')$ for all $\rhd' \in \mathcal{R}$, where $\blacktriangleleft'$

denotes the inverse ranking to $\rhd'$. Finally, since $\rhd$ maximizes $\sum_{\succ\in\mathcal{R}}(s(u(\succ,\rhd))-1)$, it follows that its inverse ranking $\blacktriangleleft$ minimizes $R(\blacktriangleleft)$. Since $|R|=1$, this implies that $R(\blacktriangleleft)\leq\frac{1}{m!}$. Hence, the only ranking with utility 0 has a weight of at most $\frac{1}{m!}<1/\binom{m}{2}$, which shows that uJR is satisfied. □

4 Results on Proportional Justified Representation

We next turn to the analysis of uPJR and sPJR. Specifically, we will shows that these proportionality axioms are implied by two notions of priceability, which we respectively call rank-priceability and pair-priceability. Based on these more structured axioms, we design several SWFs that satisfy uPJR and sPJR. Intriguingly, all of these SWFs rely on the idea of sequentially choosing candidates based on their Borda score and only aim at slightly different objectives. Moreover, we will show that each of our SWFs guarantees a high average utility to every subprofile. For space reasons, we defer the proofs of most of our results to the appendix.

4.1 Rank-priceability and the Proportional Sequential Borda rule

In this section, we will present our first SWF that satisfies uPJR, namely the Proportional Sequential Borda rule (PSB). To this end, we will discuss a more structured fairness condition called rank-priceability and show that this condition implies uPJR. Based on this result, we will then prove that PSB satisfies uPJR. Furthermore, we will show that PSB guarantees an average utility of at least $\frac{|S|}{4}\binom{m}{2}-\frac{3}{16}$ to every subprofile S.

We start by introducing rank-priceability, which is inspired by the concept of priceability studied in approval-based committee voting and participatory budgeting [e.g., 8, 37]. In these settings, voters report approval ballots over costly candidates and we need to choose a subset of candidates subject to a committee size or a budget constraint. The idea of priceability is that it should be possible to explain the outcome by a payment scheme from the voters to the chosen candidates. In more detail, a set of candidates W is called priceable if there is a virtual budget B that is uniformly distributed among the voters and a payment scheme that satisfies the following conditions:

(1) Voters only spend their share of the budget on their approved candidates in W.
(2) The total budget spent on each candidate in W is equal to its cost.
(3) The unspent budget of any group of voters is not enough to pay for a commonly approved candidate outside of W.

We next aim to transfer this axiom to rank aggregation. To this end, we assume that candidates will be bought sequentially and we update the cost of candidates and the payment willingness of rankings in each step. In more detail, in every step, the cost of a candidate will be the maximal utility it can generate for a ranking and the payment willingness of a ranking will be the additional utility it obtains by assigning the considered candidate to the next position in the ranking. To make this more formal, let $\rhd=x_1,\dots,x_m$ denote an arbitrary ranking. If we place x_i in the i-th position of the output ranking after $x_1,\dots,x_{i-1}$ have been put at positions $1,\dots,i-1$, we generate a utility of $u(\succ,x_i,\{x_i,\dots,x_m\})$ for every input ranking $\succ$. We thus require that no ranking pays more than $u(\succ,x_i,\{x_i,\dots,x_m\})$ for candidate x_i. Moreover, since $u(\succ,x_i,\{x_i,\dots,x_m\})\leq m-i$ for all $\succ\in\mathcal{R}$ and $u(\rhd,x_i,\{x_i,\dots,x_m\})=m-i$, we set the cost of x_i to $m-i$. Consequently, the total cost of all candidates is $\sum_{i=1}^{m}m-i=\binom{m}{2}$. Finally, because there is no counterpart to Condition (3) of priceability in rank aggregation, we will fix the budget to $\binom{m}{2}$ and require that the total unspent budget is less than 1. As a consequence, it may not be possible to pay for all candidates, so we use the costs of the candidates only as upper bounds. Hence, our condition may be dubbed "approximate perfect priceability", because the total budget perfectly matches the total cost of the ranking but we may not be able to cover the cost of all candidates. Formalizing these ideas results in the following condition, which we call rank-priceability.

Definition 4 (Rank-Priceability)**.** A ranking $\rhd = x_1, \dots, x_m$ is *rank-priceable* for a profile R if there is a payment function $\pi : \mathcal{R} \times C \to \mathbb{R}$ such that

(1) $0 \leq \pi(\succ, x_i) \leq u(\succ, x_i, \{x_i, \dots, x_m\})$ for all $\succ \in \mathcal{R}$ and $x_i \in C$,
(2) $\sum_{x_i \in X} \pi(\succ, x_i) \leq \binom{m}{2} \cdot R(\succ)$,
(3) $\sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) \leq m - i$ for all $i \in \{1, \dots, m\}$, and
(4) $\sum_{\succ \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ, x_i) > \binom{m}{2} - 1$.

As usual, an SWF f is rank-priceable if $f(R)$ satisfies this condition for every profile R.

We will next show that rank-priceability implies uPJR, thereby transferring one of the central results of approval-based committee voting to rank aggregation [8, 37].

Proposition 3. *If a ranking is rank-priceable for a profile, it also satisfies uPJR.*

Proof. Assume for contradiction that there is a profile R and ranking $\rhd = x_1, \dots, x_m$ such that $\rhd$ satisfies rank-priceability but not uPJR. Since $\rhd$ fails uPJR, there is an input ranking $\succ$ and an integer $\ell \in \mathbb{N}$ such that $R(\succ) \geq \ell / \binom{m}{2}$ but $u(\succ, \rhd) < \ell$. Since both $u(\succ, \rhd)$ and ℓ are integers, this means that $u(\succ, \rhd) \leq \ell - 1$. Next, let π denote a payment scheme verifying the rank-priceability of $\rhd$. By Condition (1), we have that $\pi(\succ', x_i) \leq u(\succ', x_i, \{x_i, \dots, x_m\})$ for all $x_i \in C$ and $\succ' \in \mathcal{R}$. Moreover, since $\sum_{i=1}^{m} u(\succ', x_i, \{x_i, \dots, x_m\}) = u(\succ', \rhd)$ for all rankings $\succ'$, we conclude that

$$\sum_{i=1}^{m} \pi(\succ, x_i) \leq \sum_{i=1}^{m} u(\succ, x_i, \{x_i, \dots, x_m\}) = u(\succ, \rhd) \leq \ell - 1 \leq R(\succ) \cdot \binom{m}{2} - 1.$$

Further, by Condition (2) of rank-priceability, we have that

$$\sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} \sum_{x_i \in C} \pi(\succ', x_i) \leq \binom{m}{2} \cdot \sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} R(\succ') = \binom{m}{2} \cdot (1 - R(\succ)).$$

By combining our previous two inequalities, we derive that

$$\sum_{\succ' \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ', x_i) \leq \binom{m}{2} R(\succ) - 1 + \binom{m}{2} (1 - R(\succ)) = \binom{m}{2} - 1.$$

However, this contradicts Condition (4) of rank-priceability. Hence, our initial assumption is wrong and $\rhd$ fails rank-priceability if it fails uPJR. □

Notably, the proof of Proposition 3 does not use the third condition of rank-priceability. Moreover, Condition (4) of this axiom can be weakened to only require $\sum_{x_i \in C} \pi(\succ, x_i) > \binom{m}{2} \cdot R(\succ) - 1$ for all $\succ \in \mathcal{R}$. When weakening Condition (4) in this way and omitting Condition (3), rank-priceability is equivalent to uPJR. We nevertheless decided to define rank-priceability based on Conditions (3) and (4) because these constraints give more guidance for the design of SWFs. We will next clarify this point with an example demonstrating the difference between uPJR and rank-priceability.

Example 1 (uPJR does not imply rank-priceability.)**.** Consider the following 6 rankings.

$$\succ_1 = y_1, y_2, y_3, x_1, x_2 \qquad \succ_2 = y_2, y_3, y_1, x_1, x_2 \qquad \succ_3 = y_3, y_1, y_2, x_1, x_2$$
$$\succ_4 = y_1, y_3, y_2, x_1, x_2 \qquad \succ_5 = y_2, y_1, y_3, x_1, x_2 \qquad \succ_6 = y_3, y_2, y_1, x_1, x_2$$

Moreover, let R denote the profile given by $R(\succ_i) = \frac{1}{6}$ for all $i \in \{1, \dots, 6\}$. uPJR requires for this profile that the output ranking $\rhd$ agrees in at least $\lfloor \frac{1}{6} \cdot \binom{5}{2} \rfloor = 1$ pair with every ranking $\succ_i$. While counterintuitive, this means that the ranking $\rhd = x_1, x_2, y_1, y_2, y_3$ satisfies uPJR as all input rankings agree that $x_1 \succ x_2$. However, this ranking is not rank-priceable: no ranking is willing to pay for x_2, so the input rankings can pay at most $4 + 2 + 1 = 7$ for x_1, y_1, and y_2. Since the total budget of our rankings is $\binom{5}{2} = 10$, a budget of 3 is remaining, thus showing that $\rhd$ is not rank-priceable.

Finally, we will introduce the Proportional Sequential Borda rule (PSB). On a high level, the idea of this rule is to repeatedly choose the candidate maximizing the Borda score, update the weights of the input rankings, and delete the Borda winner from the profile. To make this more formal, we assume for every step $i \in \{1, \dots, m\}$ of PSB that each ranking $\succ$ has a budget $b_i(\succ) \in \mathbb{R}_{\geq 0}$ and that there is a set of remaining candidates X_i. In the first round, it holds that $X_1 = C$ and $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ for all $\succ \in \mathcal{R}$, where R denotes the input profile. Now, in each round i, we choose the candidate x^* that maximizes the Borda score (or utilitarian welfare) $U(b_i, x, X_i) = \sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u(\succ, x, X_i)$ among all candidates x_i. Next, we place this candidate at the i-th position of the output ranking and remove it from the set of available candidates (i.e., $X_{i+1} = X_i \setminus \{x^*\}$). Furthermore, we assume that the cost of the i-th candidate is $m - i$ and, if possible, each ranking will pay a share of this cost that is proportional to its contribution to the Borda score. More formally, each ranking $\succ$ will pay either $\frac{(m-i) \cdot u(\succ, x^*, X_i) \cdot b_i(\succ)}{U(b_i, x^*, X_i)}$ or its remaining budget $b_i(\succ)$ if the proportional contribution exceeds $b_i(\succ)$. Hence, we set $b_{i+1}(\succ) = b_i(\succ) - \min(\frac{(m-i) \cdot u(\succ, x^*, X_i) \cdot b_i(\succ)}{U(b_i, x^*, X_i)}, b_i(\succ))$ for each ranking $\succ \in \mathcal{R}$. After defining X_{i+1} and the budgets $b_{i+1}(\succ)$, PSB continues with the next round until all candidates are placed in the output ranking. We will next consider an example to illustrate how the Proportional Sequential Borda rule works.

Example 2 (The Proportional Sequential Borda rule)**.** Let $\succ_1 = x_1, x_2, x_3, x_4, x_5$ and $\succ_2 = x_4, x_5, x_1, x_3, x_2$ denote two rankings and let R be the profile given by $R(\succ_1) = 0.6$ and $R(\succ_2) = 0.4$. For this profile, PSB will choose the ranking $\rhd = x_1, x_4, x_2, x_5, x_3$, which is witnessed by the following sequence of budgets and profiles.

$$
\begin{array}{cc} 6 & 4 \\ \hline x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_1 \\ x_4 & x_3 \\ x_5 & x_2 \end{array}
\quad\Longrightarrow\quad
\begin{array}{cc} 3 & 3 \\ \hline x_2 & x_4 \\ x_3 & x_5 \\ x_4 & x_3 \\ x_5 & x_2 \end{array}
\quad\Longrightarrow\quad
\begin{array}{cc} \frac{9}{4} & \frac{3}{4} \\ \hline x_2 & x_5 \\ x_3 & x_3 \\ x_5 & x_2 \end{array}
\quad\Longrightarrow\quad
\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \\ \hline x_3 & x_5 \\ x_5 & x_3 \end{array}
$$

On the left, we show the initial profile R, where the rankings are weighed by the budgets $b_1(\succ_1) = R(\succ_1) \cdot \binom{5}{2} = 6$ and $b_1(\succ_2) = R(\succ_2) \cdot \binom{5}{2} = 4$. Candidate x_1 maximizes the Borda score in this profile as $U(b_1, x_1, \{x_1, \dots, x_5\}) = 6 \cdot 4 + 4 \cdot 2 = 32$. Consequently, $\succ_1$ pays $\frac{4}{32} \cdot 6 \cdot 4 = 3$ and $\succ_2$ pays $\frac{4}{32} \cdot 4 \cdot 2 = 1$, which means that the new budgets are $b_2(\succ_1) = b_2(\succ_2) = 3$. We moreover remove x_1 from the profile as $X_2 = C \setminus \{x_1\}$. In the second step, x_4 maximizes the total Borda score with $U(b_2, x_4, \{x_2, \dots, x_5\}) = 12$, so $\succ_1$ pays $\frac{3}{12} \cdot 3 \cdot 1 = \frac{3}{4}$ and $\succ_2$ pays $\frac{3}{12} \cdot 3 \cdot 3 = \frac{9}{4}$. Consequently, the new budgets are given by $b_3(\succ_1) = \frac{9}{4}$ and $b_3(\succ_2) = \frac{3}{4}$ and x_4 is removed. In the third step, x_2 maximizes the total Borda score with $U(b_3, x_2, \{x_2, x_3, x_5\}) = \frac{9}{2}$ and $\succ_1$ pays the total cost of 2 since $u(\succ_2, x_2, \{x_2, x_3, x_5\}) = 0$. Hence, the budgets in the fourth step are $b_4(\succ_1) = \frac{1}{4}$ and $b_4(\succ_2) = \frac{3}{4}$. Finally, PSB now picks x_5 and $\succ_2$ will pay its remaining budget of $\frac{3}{4}$.

We note that the total leftover budget in Example 2 is only $\frac{1}{4}$, which implies that PSB is rank-priceable in this example. We will next show that this holds in general, i.e., the Proportional Sequential Borda rule satisfies rank-priceability and therefore also uPJR.

Theorem 1. *The Proportional Sequential Borda rule satisfies rank-priceability.*

Proof. Fix a profile R and let $\rhd = x_1, \dots, x_m$ be the ranking chosen by PSB. Moreover, we let $b_i(\succ)$ denote the budgets used during the computation of PSB. We will show that the payment scheme π defined by

$$\pi(\succ, x_i) = b_i(\succ) - b_{i+1}(\succ) = \min\left(\frac{(m-i)\cdot u(\succ, x_i, X_i)\cdot b_i(\succ)}{U(b_i, x_i, X_i)}, b_i(\succ)\right)$$

for all $i \in \{1, \dots, m-1\}$ and all $\succ \in \mathcal{R}$ satisfies the conditions of rank-priceability.

Condition (1): Fix a step $i \in \{1, \dots, m-1\}$ and a ranking $\succ$ with $b_i(\succ) > 0$. It holds that $U(b_i, x_i, \{x_i, \dots, x_m\}) \geq (m-i)b_i(\succ)$ because x_i maximizes the Borda score and $(m-i)b_i(\succ)$ is a lower bound for the Borda score of the top-ranked candidate of $\succ$. Hence, we derive the following inequality, which shows Condition (1).

$$\pi(\succ, x_i) \leq \frac{(m-i)u(\succ, x_i, X_i)b_i(\succ)}{U(b_i, x_i, X_i)} \leq \frac{(m-i)u(\succ, x_i, X_i)b_i(\succ)}{(m-i)b_i(\succ)} = u(\succ, x_i, \{x_i, \dots, x_m\})$$

Condition (2): For this condition, we note for all $\succ \in \mathcal{R}$ that $b_1(\succ) = R(\succ)\cdot\binom{m}{2}$ and that $b_m(\succ) \geq 0$ because $\pi(\succ, x_i) \leq b_i(\succ)$ for all $i \in \{1, \dots, m-1\}$. Hence, $\sum_{i=1}^m \pi(\succ, x_i) \leq R(\succ)\cdot\binom{m}{2}$ for all $\succ \in \mathcal{R}$ and Condition (2) of rank-priceability holds.

Condition (3): Condition (3) is satisfied because it holds for every $i \in \{1, \dots, m-1\}$ that

$$\sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) \leq \sum_{\succ \in \mathcal{R}} \frac{(m-i)\cdot u(\succ, x, X_i)\cdot b_i(\succ)}{U(b_i, x, X_i)} = m-1.$$

Condition (4): For Condition (4), we will first show that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$ for all $i \geq m-2$. Clearly, this is true if $i = 1$ because $\sum_{\succ \in \mathcal{R}} b_1(\succ) = \binom{m}{2}\sum_{\succ \in \mathcal{R}} R(\succ) = \frac{(m-1)(m)}{2}$ by definition. Next, we inductively assume that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$ for some $i \in \{1, \dots, m-3\}$ and we let $X_i = \{x_i, \dots, x_m\}$. Since there are $m-i+1$ candidates in X_i, it follows that

$$\sum_{x \in X_i}\sum_{\succ \in \mathcal{R}} b_i(\succ)u(\succ, x, X_i) = \sum_{\succ \in \mathcal{R}} b_i(\succ)\sum_{j=0}^{m-i} j = \left(\frac{(m-i)(m-i+1)}{2}\right)^2.$$

Since candidate x_i maximizes the Borda score with respect to b_i and X_i, we conclude that $U(b_i, x_i, X_i) \geq \frac{1}{m-i+1}\sum_{x \in X_i} U(b_i, x, X_i) = \frac{(m-i)^2(m-i+1)}{4}$. This implies for all $\succ \in \mathcal{R}$ that $\frac{(m-i)u(\succ, x, X_i)b_i(\succ)}{U(b_i, x_i, X_i)} \leq \frac{4u(\succ, x_i, X_i)b_i(\succ)}{(m-i)(m-i+1)}$. Finally, it holds that $m-i+1 \geq 4$ as $m-i \geq 3$ and $u(\succ, x_i, X_i) \leq m-i$ by definition. We thus infer that $\frac{(m-i)u(\succ, x, X_i)b_i(\succ)}{U(b_i, x_i, X_i)} \leq b_i(\succ)$ for all $\succ \in \mathcal{R}$. Since $\sum_{\succ \in \mathcal{R}} \frac{(m-i)u(\succ, x, X_i)b_i(\succ)}{U(b_i, x_i, X_i)} = m-i$, the total budget in round $i+1$ is

$$\begin{aligned}\sum_{\succ \in \mathcal{R}} b_{i+1}(\succ) &= \sum_{\succ \in \mathcal{R}} b_i(\succ) - \frac{(m-i)\cdot u(\succ, x, X_i)\cdot b_i(\succ)}{U(b_i, x_i, X_i)} \\ &= \frac{(m-i)(m-i+1)}{2} - (m-i) \\ &= \frac{(m-i)(m-i-1)}{2}.\end{aligned}$$

For $i = m-2$, this reasoning shows that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2} = 3$. Furthermore, if $i = m-2$, we are left with the candidates $X_{m-2} = \{x_{m-2}, x_{m-1}, x_m\}$ and we know that x_{m-2} maximizes the Borda score with respect to b_{m-2}. Now, if

$$\min\left(\frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})}, b_{m-2}(\succ)\right) = \frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})}$$

for all $\succ \in \mathcal{R}$, our previous reasoning shows that we decrease the total budget by 2. Next, assume that

$$\min(\frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})}, b_{m-2}(\succ)) = b_{m-2}(\succ)$$

for some ranking $\succ$. In this case, we first note that candidate x_{m-2} has a Borda score of at least $\frac{2\cdot 2\cdot 3}{4} = 3$. Hence, $\frac{2u(\succ,x_{m-2},X_{m-2})b_{m-2}(\succ)}{U(b_{m-2},x_{m-2},X_{m-2})} \leq \frac{2}{3}b_{m-2}(\succ)$ for all $\succ \in \mathcal{R}$ with $u(\succ, x_{m-2}, X_{m-2}) \leq 1$. Consequently, $\min(\frac{2u(\succ,x_{m-2},X_{m-2})b_{m-2}(\succ)}{U(b_{m-2},x_{m-2},X_{m-2})}, b_{m-2}(\succ)) = b_{m-2}(\succ)$ is only possible if $u(\succ, x_{m-2}, X_{m-2}) = 2$. Now, let Y denote the set of rankings such that $\succ \in Y$ if and only if $u(\succ, x_{m-2}, X_{m-2}) = 2$. By our analysis so far, it holds that

$$\begin{aligned}
&\sum_{\succ \in \mathcal{R}} \min\left(\frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2},\ X_{m-2})}, b_{m-2}(\succ)\right) \\
&= \sum_{\succ \in \mathcal{R}} \frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})} - \sum_{\succ \in Y}\left(\frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})} - b_{m-2}(\succ)\right) \\
&= 2 - \left(\frac{4}{U(b_{m-2}, x_{m-2}, X_{m-2})} - 1\right)\sum_{\succ \in Y} b_{m-2}(\succ).
\end{aligned}$$

By the assumption that $\min(\frac{2u(\succ,x_{m-2},X_{m-2})\cdot b_{m-2}(\succ)}{U(b_{m-2},x_{m-2},X_{m-2})}, b_{m-2}(\succ)) = b_{m-2}(\succ)$ for $\succ \in Y$, we derive that $(\frac{4}{U(b_{m-2},x_{m-2},X_{m-2})} - 1) \geq 0$. Hence, the above term is minimized if $\sum_{\succ \in Y} b_{m-2}(\succ)$ is maximized. Because $\sum_{\succ \in Y} b_{m-2}(\succ) \leq \frac{U(b_{m-2},x_{m-2},X_{m-2})}{2}$ and $U(b_{m-2}, x_{m-2}, X_{m-2}) \geq 3$, we get that

$$\begin{aligned}
&\sum_{\succ \in \mathcal{R}} \min\left(\frac{2u(\succ, x_{m-2}, X_{m-2})b_{m-2}(\succ)}{U(b_{m-2}, x_{m-2}, X_{m-2})}, b_{m-2}(\succ)\right) \\
&\geq 2 - (\frac{4}{U(b_{m-2}, x_{m-2}, X_{m-2})} - 1) \cdot \frac{U(b_{m-2}, x_{m-2}, X_{m-2})}{2} \\
&\geq 2 - 2 + \frac{U(b_{m-2}, x_{m-2}, X_{m-2})}{2} \\
&\geq \frac{3}{2}.
\end{aligned}$$

In summary, we conclude that the total budget decreases by at least $\frac{3}{2}$, so the total remaining budget for the last round is at most 1.5. In this round, only the candidates $X_{m-1} = \{x_{m-1}, x_m\}$ are left, so it holds that $u(\succ, x_{m-1}, X_{m-1}) \in \{0, 1\}$. If $U(b_{m-1}, x_{m-1}, X_{m-1}) \geq 1$, this means that $\min(\frac{u(\succ,x_{m-1},X_{m-1})b_{m-1}(\succ)}{U(b_{m-1},x_{m-1},X_{m-1})}, b_{m-1}(\succ)) = \frac{u(\succ,x_{m-1},X_{m-1})b_{m-1}(\succ)}{U(b_{m-1},x_{m-1},X_{m-1})}$ and we decrease the total budget by at least 1. Hence, the total remaining budget is at most 0.5, which proves Condition (4) of rank-priceability in this case. By contrast, if $U(b_{m-1}, x_{m-1}, X_{m-1}) < 1$, each ranking with $u(\succ, x_{m-1}, X_{m-1}) = 1$ will contribute its complete budget. Since x_{m-1} maximizes the Borda score, we derive that $\sum_{\succ \in \mathcal{R}\colon x_{m-1} \succ x_m} b_{m-1}(\succ) \geq \sum_{\succ \in \mathcal{R}\colon x_m \succ x_{m-1}} b_{m-1}(\succ)$, so the total remaining budget is reduced by at least half. So, the total remaining budget is at most 0.75 and Condition (4) holds. □

As a second fairness property, we will also analyze the average utility that PSB guarantees to subprofiles, as a function of the size of the subprofile. Intuitively, a proportional SWF should guarantee to each subprofile S a fraction of the total utility $\binom{m}{2}$ that is at least linear in $|S|$. We show that PSB meets this condition as every subprofile S is guaranteed an average utility of at least $\frac{|S|}{4} - \frac{3}{16}$ by our SWF. We note that Lederer et al. [31] have computed an analogous bound for the Squared Kemeny rule, which is, however, sub-linear and trivial if $|S| \leq \frac{1}{4}$. The proof of this result can be found in Appendix B.

Theorem 2. *Let R be a profile on m candidates and $\rhd$ = $\mathsf{PSB}(R)$ be the ranking chosen by the Proportional Sequential Borda rule. It holds for every subprofile S of R that*

$$\frac{1}{|S|}\sum_{\succ\in\mathcal{R}} S(\succ)u(\succ,\rhd) \geq \binom{m}{2}\cdot\frac{|S|}{4}-\frac{3}{16}.$$

4.2 The Ranked Method of Equal Shares

We will next discuss our first variant of the Proportional Sequential Borda rule, which aims at finding more utilitarian rankings while still satisfying uPJR. Specifically, we will introduce the Ranked Method of Equal Shares (RMES) which simultaneously satisfies uPJR and guarantees to pick the first $\lfloor\frac{m}{4}\rfloor$ candidates in a highly utilitarian way.

Since the Ranked Method of Equal Shares is closely related to the Method of Equal Shares [36, 37], we will outline this method first. Just as PSB, the Method of Equal Shares uniformly distributes a budget to the voters who use it to buy costly candidates. In more detail, let $b_i(j)$ denote the remaining budget of voter j in the i-th round, $u(j,x)$ the utility of voter j for candidate x, and $c(x)$ the cost of candidate x. The Method of Equal Shares chooses in the i-th round the candidate x^* that minimizes the value ρ for which $\sum_{j\in N}\min(\rho\cdot u(x^*,j),b_i(j))=c(x^*)$. Furthermore, after buying candidate x^*, the budget of each voter is decreased by his contribution to the cost of x^*, i.e., $b_{i+1}(j)=b_i(j)-\min(\rho\cdot u(x^*,j),b_i(j))$. Based on the ideas of the last section, this approach can be easily extended to rank aggregation. Specifically, we will assume that the cost of each candidate in the i-th round is $m-i$. Then, the Method of Equal Shares chooses in the i-th round the candidate x_i that minimizes the price ρ such that $\sum_{\succ\in\mathcal{R}}\min(\rho\cdot b_1(\succ)\cdot u(\succ,x_i,\{x_i,\dots,x_m\}),b_i(\succ))=m-i$. However, it turns out that this method fails rank-priceability as a ranking may pay more for a candidate than the utility it obtains.[4] We will thus include the term $u(\succ,x_i,\{x_i,\dots,x_m\})$ as a third argument of the minimum.

We now formally define the *Ranked Method of Equal Shares (*RMES*)*, which iteratively selects candidates based on the budgets $b_i(\succ)$ of the input rankings and the set of remaining candidates X_i. As for PSB, it holds in the first round that $b_1(\succ)=R(\succ)\cdot\binom{m}{2}$ and $X_1=C$. Now, in each round $i\in\{1,\dots,m-2\}$, RMES identifies the candidate $x_i\in X_i$ that minimizes the value ρ_i such that

$$\sum_{\succ\in\mathcal{R}}\min(\rho_i\cdot b_1(\succ)\cdot u(\succ,x_i,X_i),b_i(\succ),u(\succ,x_i,X_i))=m-i.$$

Then, we place this candidate at the i-th position of the output ranking, remove x_i from the active candidates, and reduce the budget of every ranking according to its contribution to the cost of x_i. More formally, we set $X_{i+1}=X_i\setminus\{x_i\}$ and $b_{i+1}(\succ)=b_i(\succ)-\min(\rho_i\cdot b_1(\succ)\cdot u(\succ,x_i,X_i),b_i(\succ),u(\succ,x_i,X_i))$ for all $\succ$. After this, we proceed with the next round. Finally, since this approach is only guaranteed to work when $|X_i|\geq 3$, we decide the order over the last two candidates by majority voting with respect to the remaining budgets: if x and y are the last active candidates, we place x ahead of y at the $m-1$-th position of the output ranking if $\sum_{\succ\in\mathcal{R}:\,x\succ y}b_{m-1}(\succ)\geq\sum_{\succ\in\mathcal{R}:\,y\succ x}b_{m-1}(\succ)$. Otherwise, we put y at the $m-1$-th position. As usual, ties can be broken arbitrarily.

[4] For an example, consider the rankings $\succ_1=x_1,\dots,x_6$ and $\succ_2=x_6,\dots,x_1$ and let R be the profile defined by $R(\succ_1)=\frac{47}{60}=\frac{11.75}{15}$ and $R(\succ_2)=\frac{13}{60}=\frac{3.25}{15}$. Since $\binom{6}{2}=15$, the initial budgets of $\succ_1$ and $\succ_2$ are $b_1(\succ_1)=R(\succ_1)\cdot 15=11.75$ and $b_1(\succ_2)=R(\succ_2)\cdot 15=3.25$. In the first two rounds, it is easy to verify that RMES chooses x_1 and x_2 and that $\succ_1$ will pay the full cost of these candidates. Hence, in the third step, we have that $b_3(\succ_1)=11.75-5-4=2.75$ and $b_3(\succ_2)=3.25$. This means that x_3 is no longer feasible because $\succ_1$ has not enough budget left to pay for this candidate and $\succ_2$ gains no utility from x_3. If we do not include the utility $u(\succ,x,X_i)$ in the minimum for computing ρ, we would thus buy x_4 for a cost per utility ratio of $\rho=\frac{3}{2\cdot 11.75+1\cdot 3.25}=\frac{12}{107}$. In turn, we infer that $\succ_1$ pays $\frac{12}{107}\cdot\frac{47}{4}\cdot 2=\frac{282}{107}\approx 2.64$ and $\succ_2$ pays $\frac{12}{107}\cdot\frac{13}{4}\cdot 1=\frac{39}{107}\approx 0.36$. However, this means that $\succ_2$ pays more than its obtained utility in this step, thus violating rank-priceability.

Example 3 (The Ranked Method of Equal Shares). Let $\succ_1 = x_1, x_2, x_3, x_4, x_5$ and $\succ_2 = x_4, x_5, x_1, x_3, x_2$ and consider the same profile R as in Example 2, i.e., $R(\succ_1) = 0.6$ and $R(\succ_2) = 0.4$. If the tie-breaking favors candidates with smaller indices, RMES chooses the ranking $\rhd = x_1, x_2, x_4, x_5, x_3$ for this profile, as verified by the following computations.

$$
\begin{array}{cc}
6\ (6) & 4\ (4) \\
\hline
x_1 & x_4 \\
x_2 & x_5 \\
x_3 & x_1 \\
x_4 & x_3 \\
x_5 & x_2
\end{array}
\implies
\begin{array}{cc}
6\ (3) & 4\ (3) \\
\hline
x_2 & x_4 \\
x_3 & x_5 \\
x_4 & x_3 \\
x_5 & x_2
\end{array}
\implies
\begin{array}{cc}
6\ (0) & 4\ (3) \\
\hline
x_3 & x_4 \\
x_4 & x_5 \\
x_5 & x_3
\end{array}
\implies
\begin{array}{cc}
6\ (0) & 4\ (1) \\
\hline
x_3 & x_5 \\
x_5 & x_3
\end{array}
$$

Here, we show the input rankings restricted to the available candidates and weighted by their initial budget $b_1(\succ)$. In brackets, we also show the remaining budget in each round. Analogous to PSB, RMES picks in the first step x_1 for a price $\rho_1 = \frac{1}{8}$, so $\succ_1$ pays $\frac{1}{8} \cdot 4 \cdot 6 = 3$ and $\succ_2$ pays $\frac{1}{8} \cdot 2 \cdot 4 = 1$. Hence, the new budgets are $b_2(\succ_1) = 3$ and $b_2(\succ_2) = 3$. In the second step, both x_2 and x_4 can be bought for a price of $\rho = \frac{1}{6} = \frac{3}{18}$. Because we assume that the tie-breaking favors x_2 to x_4, we pick x_2 next. Consequently, $\succ_1$ pays $\frac{1}{6} \cdot 3 \cdot 6 = 3$ and $\succ_2$ pays $\frac{1}{6} \cdot 0 \cdot 4 = 0$, which means that $b_3(\succ_1) = 0$ and $b_3(\succ_2) = 3$. From here on, RMES picks the candidates according to $\succ_2$ as $\succ_1$ has no budget left.

We note that in this example, there is always a candidate x_i that can be bought for a finite price. We next show that this observation holds in general as RMES is well-defined. Moreover, we will also prove that RMES satisfies rank-priceability and thus uPJR.

Theorem 3. RMES *is well-defined and satisfies rank-priceability.*

While Theorem 3 establishes that RMES is a proportional SWF, we will next show that it is still rather utilitarian. Specifically, we will prove that the first $\lfloor \frac{m}{4} \rfloor$ candidates of this rule are chosen only based on the Borda scores with respect to the initial weights. Put differently, RMES agrees for roughly the first quarter of the candidates with the highly utilitarian ranking obtained by repeatedly placing the Borda winner in the next available position of the output ranking and removing it from the input profile. Moreover, these candidates determine roughly $\frac{7}{16}$ of all pairwise comparisons, thus showing that a significant portion of the total utility is assigned in a utilitarian way.

Proposition 4. *Fix a profile R on m candidates and let $\rhd = x_1, \dots, x_m$ denote the ranking chosen by* RMES. *It holds for all $i \in \{1, \dots, \lfloor \frac{m}{4} \rfloor\}$ that $x_i = \arg\max_{x \in \{x_i, \dots, x_m\}} U(b_1, x, \{x_i, \dots, x_m\})$.*

As our last result on RMES and analogous to Theorem 2, we will present a lower bound on the average utility of subprofiles ensured by RMES. Interestingly, we will show that Proposition 4 implies a slightly better guarantee for large subprofiles compared to PSB because, roughly, this result entails a lower cost per utility ratio in the first steps.

Theorem 4. *Let R be a profile on $m \geq 4$ candidates, $\rhd = \text{RMES}(R)$, and define $\xi = \binom{m - \lfloor \frac{m}{4} \rfloor}{2}$. It holds for every subprofile S of R that*

$$
\frac{1}{|S|} \sum_{\succ \in \mathcal{R}} S(\succ) u(\succ, \rhd) \geq
\begin{cases}
\binom{m}{2} \cdot \frac{|S|}{4} - \frac{1}{8} & \text{if } \binom{m}{2}|S| - 0.5 \leq \xi \\
\frac{1}{2} \cdot \binom{m}{2} \cdot \left(1 - \frac{\xi}{\binom{m}{2}|S|}\right) + \frac{\xi+1}{4} \cdot \frac{\xi}{\binom{m}{2}|S|} - \frac{1}{4|S|} & \text{if } \binom{m}{2}|S| - 0.5 > \xi.
\end{cases}
$$

4.3 Pair-Priceability and the Flow-adjusting Borda Rule

As our second variant of PSB, we will discuss the Flow-adjusting Borda rule (FB), which satisfies sPJR. To motivate this rule, we first show that PSB and RMES fail this stronger fairness condition.

Example 4 (PSB and RMES fail PJR)**.** We will consider a profile with 25 candidates $\{y, x_1, \dots, x_4, z_1, \dots, z_{20}\}$ and the following 8 rankings:

$$\succ_1 = x_1, x_2, x_3, x_4, y, z_1, \dots, z_{20} \qquad \succ_2 = x_2, x_3, x_4, x_1, y, z_1, \dots, z_{20}$$
$$\succ_3 = x_3, x_4, x_1, x_2, y, z_1, \dots, z_{20} \qquad \succ_4 = x_4, x_1, x_2, x_3, y, z_1, \dots, z_{20}$$
$$\succ_5 = y, z_{20}, \dots, z_1, x_4, x_3, x_2, x_1 \qquad \succ_6 = y, z_{20}, \dots, z_1, x_3, x_2, x_1, x_4$$
$$\succ_7 = y, z_{20}, \dots, z_1, x_2, x_1, x_4, x_3 \qquad \succ_8 = y, z_{20}, \dots, z_1, x_1, x_4, x_3, x_2$$

Less formally, our ranking can be partitioned into 2 groups: the rankings $\succ_1, \dots, \succ_4$ rank all x_i ahead of y ahead of all z_j, order the candidates z_j in increasing order of their indices, and the candidates x_i are arranged cyclic within these rankings. Conversely, the rankings $\succ_5, \dots, \succ_8$ rank y ahead of all z_j ahead of all x_i, rank the candidates z_j in decreasing order of their indices, and the candidates x_i are also arranged in a cycle within these rankings. We next note that $\binom{25}{2} = 300$ and we define R as the ranking such that $R(\succ_i) = \frac{67.5}{300}$ for all $i \in \{1, \dots, 4\}$ and $R(\succ_i) = \frac{7.5}{300}$ for all $i \in \{5, \dots, 8\}$. In particular, this means that PJR requires that the output ranking $\rhd$ chooses $67.5 \cdot 4 = 270$ pairwise comparisons from the union of $\succ_1, \dots, \succ_4$. However, we have shown with the help of a computer that, under suitable tie-breaking in the first 5 steps, PSB and RMES choose the following rankings:

$$\mathsf{PSB}(R) = \rhd_{\mathsf{PSB}} = y, x_1, x_2, x_3, x_4, z_1, \dots, z_{10}, z_{20}, z_{11}, z_{19}, z_{18}, z_{12}, z_{13}, z_{17}, z_{16}, z_{14}, z_{15}$$
$$\mathsf{RMES}(R)) = \rhd_{\mathsf{RMES}} = y, x_1, x_2, x_3, x_4, z_1, \dots, z_{11}, z_{19}, z_{20}, \dots, z_{12}.$$

It can be verified that both $\rhd_{\mathsf{PSB}}$ and $\rhd_{\mathsf{RMES}}$ only agree with 269 pairwise comparisons in the union of $\succ_1, \dots, \succ_4$, thus witnessing a violation of PJR. While the full computations for these output rankings is tedious, we note that for both rules, the central "mistake" happens in the first round. In this round, each candidate $c \in \{x_1, \dots, x_4, y\}$ has a Borda score of $U(b_1, c, C) = 6120$ (where $b_1(\succ) = R(\succ) \cdot 300$). Hence, under suitable tie-breaking, both rules pick y first. Moreover, for both rules, each ranking $\succ_i$ with $i \in \{1, \dots, 4\}$ pays $\frac{24}{6120} \cdot 20 \cdot 67.5 = \frac{90}{17}$ and the rankings $\succ_i$ with $i \in \{5, \dots, 8\}$ each pay $\frac{24}{6120} \cdot 24 \cdot 7.5 = \frac{12}{17}$. However, this means that the rankings $\succ_1, \dots, \succ_4$ pay in total $\frac{360}{17} \approx 21.18$, even though each of these rankings only obtains a utility of 20 from placing y first. Put differently, these rankings pay as a group more than their obtained utility, so they cannot afford enough further candidates to get the utility they deserve.

We note that the problem in Example 4 can also be seen as a flaw in the definition of rank-priceability: this axiom only precludes that individual rankings spend more on a candidate than the utility they obtain, but this guarantee does not extend to groups. To design SWFs that satisfy sPJR, we will therefore present a refined version of rank-priceability called pair-priceability. The idea of this axiom is to view the output ranking $\rhd$ as the set of pairs $A(\rhd) = \{(x, y) \in C^2 \colon x \rhd y\}$ and that every pair of this set needs to be bought for a price of 1 by the input rankings.

Definition 5 (Pair-Priceability)**.** A ranking $\rhd = x_1, \dots, x_m$ is *pair-priceable* for a profile R if there is a payment function $\pi : \mathcal{R} \times A(\rhd) \to [0, 1]$ such that

(1) $\pi(\succ, (x_i, x_j)) \leq u(\succ, x_i, \{x_i, x_j\})$ for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in A(\rhd)$.
(2) $\sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) \leq \binom{m}{2} \cdot R(\succ)$ for all $\succ \in \mathcal{R}$.
(3) $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) \leq 1$ for all $(x_i, x_j) \in A(\rhd)$.
(4) $\sum_{\succ \in \mathcal{R}} \sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) > \binom{m}{2} - 1$.

Pair-priceability differs from rank-priceability only in that Conditions (1) and (3) are formulated for pairs of candidates rather than for candidates. For instance, Condition (1) now states that a ranking $\succ$ is only allowed to pay for a pair of candidates (x_i, x_j) if $x_i \succ x_j$. Hence, pair-priceability requires a more fine-grained payment scheme than rank-priceability. Further, we note that rank-priceability rules out the problem observed in Example 4: the rankings $\succ_1, \dots, \succ_4$ in this example can pay at most 20 for the pairwise comparisons including y because they all rank y only ahead of $z_1, \dots, z_{20}$. More generally, we will next show that pair-priceability implies sPJR and that pair-priceable rankings are guaranteed to exist. Curiously, the proof that pair-priceable rankings exist is driven by the ranking-matching lemma, one of the central tools in the analysis of the metric distortion of voting rules [21, 23, 24].

Proposition 5. *The following claims are true:*

(1) If a ranking is pair-priceable for a profile, it also satisfies sPJR.
(2) For every profile, there is a pair-priceable ranking.

Since the proof of Claim (2) of Proposition 5 is constructive, it directly yields an SWF that satisfies rank-priceability. In particular, it is possible to define SWFs satisfying pair-priceability by adapting voting rules designed in the context of metric distortion to rank aggregation. For instance, one can combine the SimulatenousVeto rule of Kizilkaya and Kempe [24] with a budgeting approach to derive an SWF that satisfies pair-priceability and thus sPJR. However, while we find this direction interesting, we leave the analysis of such SWFs for future work. Instead, we will suggest another method based on Borda scores that is pair-priceable.

Specifically, we will now discuss the *Flow-adjusting Borda rule* (FB). The idea of this rule is similar to PSB: in each round, we will add the Borda winner with respect to the current budgets to the output ranking, decrease the budgets of the rankings, and remove the Borda winner from consideration. To make this more formal, we denote again by $b_i(\succ)$ the budget of ranking $\succ$ in the i-th round and by X_i the remaining candidates. Just as for PSB, we have that $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ and $X_1 = C$, where R is the input profile. For each round i, we will then choose the candidate $x^* = \arg\max_{x \in X_i} U(b_i, x, X_i)$ maximizing the Borda score, place it at the i-th position of the output ranking, and set $X_{i+1} = X_i \setminus \{x^*\}$. However, in contrast to PSB, FB determines the payments of the rankings based on a maximum flow in the following flow network $G_{x^*} = (V, E, c)$.[5]

- The set of vertices V contains a source s, a ranking vertex $v_\succ$ for every $\succ \in \mathcal{R}$, a candidate vertex v_y for every $y \in X_i \setminus \{x^*\}$, and a sink t.
- For every ranking $\succ$, there is an edge from the source s to the ranking vertex $v_\succ$ with a capacity equal to the remaining budget of $\succ$, i.e., $c(s, v_\succ) = b_i(\succ)$.
- For every ranking $\succ$ and every candidate $y \in X_i \setminus \{x^*\}$ with $x^* \succ y$, there is an edge from $v_\succ$ to v_y with unbounded capacity.
- For every candidate y, there is an edge from the candidate vertex v_y to the sink t with capacity $c(v_y, t) = 1$.

Now, let f denote an arbitrary maximum flow in G_{x^*} that optimizes the maximum cost per utility ratio of an input ranking, i.e., that minimizes $\max_{\succ \in \mathcal{R}} \frac{f(s, v_\succ)}{b_i(\succ) u(\succ, x^*, X_i)}$ (where we assume for simplicity that $\frac{0}{0} = 0$). After determining this flow, we set $b_{i+1}(\succ) = b_i(\succ) - f(s, v_\succ)$ for every ranking $\succ$ and proceed with the next round. By this definition, FB only augments PSB by using a more sophisticated payment scheme. Even more, if possible, every ranking pays in FB the same

[5] We recall here some basics for the maximum flow problem. A flow network $G = (V, E, c)$ is a capacitated directed graph where $c : E \to \mathbb{R}_{\geq 0}$ specifies the capacity of every edge and V contains two designated vertices s and t called source and sink. A flow in such a network is a function $f : E \to \mathbb{R}_{\geq 0}$ such that *(i)* $f(e) \leq c(e)$ for all $e \in E$ (capacity constraint) and *(ii)* $\sum_{(u,v) \in E} f(u, v) = \sum_{(v,w) \in E} f(v, w)$ for all $v \in V \setminus \{s, t\}$ (flow conservation). The value of a flow f is the net outflow of the source s, i.e., $\sum_{(s,v) \in E} f(s, v) - \sum_{(v,s) \in E} f(v, s)$. Finally, a maximum flow is a flow with maximum value. To simplify notation, we will often write $f(u, v) = 0$ if $(u, v) \notin E$.

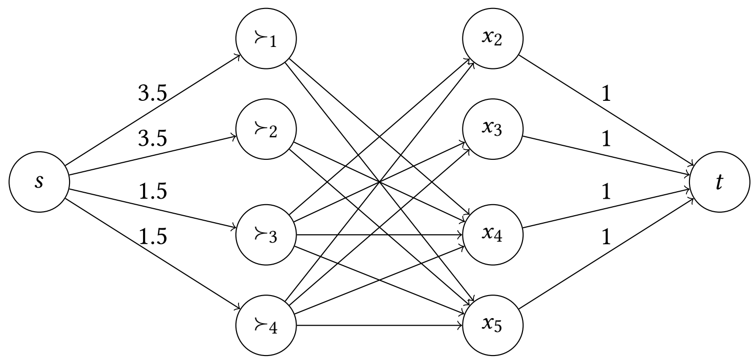

Fig. 1. The flow network G_{x_1} used for the first step of FB for the profile R shown in Example 5.

amount as in PSB because $\max_{\succ \in \mathcal{R}} \frac{f(s,v_\succ)}{b_i(\succ)u(\succ,x^*,X_i)}$ is minimized if $f(s,v_\succ) = \frac{v(f)b_i(\succ)u(\succ,x^*,X_i)}{U(b_i,x^*,X_i)}$ for all $\succ \in \mathcal{R}$ (where $v(f)$ denotes the value of f).

Example 5 (The Flow-adjusting Borda rule)**.** We consider the following four rankings.

$$\succ_1 = x_2, x_3, x_1, x_4, x_5 \qquad \succ_2 = x_3, x_2, x_1, x_4, x_5$$
$$\succ_3 = x_1, x_4, x_5, x_2, x_3 \qquad \succ_4 = x_1, x_4, x_5, x_3, x_2$$

Moreover, let R be the profile given by $R(\succ_1) = R(\succ_2) = \frac{7}{20}$ and $R(\succ_3) = R(\succ_4) = \frac{3}{20}$. Assuming that ties are broken in favor of candidates with smaller indices, FB chooses the ranking $\rhd = x_1, x_2, x_3, x_4, x_5$ for this profile, whereas PSB chooses $\rhd = x_1, x_2, x_4, x_3, x_5$. The computation for FB can be verified based on the following sequence of profiles.

$$\begin{array}{cccc} \frac{7}{2} & \frac{7}{2} & \frac{3}{2} & \frac{3}{2} \\ \hline x_2 & x_3 & x_1 & x_1 \\ x_3 & x_2 & x_4 & x_4 \\ x_1 & x_1 & x_5 & x_5 \\ x_4 & x_4 & x_2 & x_3 \\ x_5 & x_5 & x_3 & x_2 \end{array} \Longrightarrow \begin{array}{cccc} \frac{5}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline x_2 & x_3 & x_4 & x_4 \\ x_3 & x_2 & x_5 & x_5 \\ x_4 & x_4 & x_2 & x_2 \\ x_5 & x_5 & x_3 & x_3 \end{array} \Longrightarrow \begin{array}{cc} \frac{55}{26} & \frac{23}{26} \\ \hline x_3 & x_4 \\ x_4 & x_5 \\ x_5 & x_3 \end{array} \Longrightarrow \begin{array}{cc} \frac{3}{26} & \frac{13}{26} \\ \hline x_4 & x_4 \\ x_5 & x_5 \end{array}$$

We show in this graphic again the rankings restricted to the available candidates and weighted by their budget in each round. Moreover, we collapsed in the third step the rankings $\succ_1$ and $\succ_2$ as well as $\succ_3$ and $\succ_4$ into single rankings. In the first round of FB, it holds for $x \in \{x_1, x_2, x_3\}$ that $U(b_1, x, C) = 26$. By our tie-breaking assumption, this means that x_1 is chosen and we need to identify a maximum flow in the network G_{x_1} shown in Figure 1. In this network, the rankings $\succ_1$ and $\succ_2$ together can pay at most 2 for x_4 and x_5 and the rankings $\succ_3$ and $\succ_4$ can pay 1 each for x_1 and x_2. Hence, the maximum flow has value 4 and it can be shown that the cost per utility ratio is minimized if each ranking pays 1. Hence, the budgets in the second step are $b_2(\succ_1) = b_2(\succ_2) = \frac{5}{2}$ and $b_2(\succ_3) = b_2(\succ_4) = \frac{1}{2}$. By contrast, in PSB, $\succ_1$ and $\succ_2$ each pay $\frac{4}{26} \cdot 2 \cdot \frac{7}{2} = \frac{28}{26}$, which is the main reason for the different outcome. Starting from the second round on, FB behaves exactly like PSB, because the payments made by PSB can be transformed into a maximum flow of the corresponding network. We hence leave the verification of the remaining steps to the reader.

We will next show that FB is pair-priceable and thus satisfies sPJR. Moreover, we note that the following statement holds regardless of the exact maximum flow chosen in the flow network G_{x_i}, i.e., it is not necessary to minimize the cost per utility ratio.

Theorem 5. *The Flow-adjusting Borda rule is pair-priceable.*

Lastly, we also examine the average utility that FB guarantees to subprofile. Specifically, we will next show that FB gives the same guarantee on the average utility of subprofiles than PSB. For the proof of the subsequent theorem, it is crucial that FB chooses the maximum flow that minimizes the cost per utility ratio in every step.

Theorem 6. *Let R be a profile on m candidates and $\rhd = \mathtt{FB}(R)$ the ranking chosen by the Flow-adjusting Borda rule. It holds for every subprofile S of R that*

$$\frac{1}{|S|}\sum_{\succ\in\mathcal{R}} S(\succ)u(\succ,\rhd) \geq \binom{m}{2}\cdot\frac{|S|}{4} - \frac{3}{16}.$$

5 Conclusion

In this paper, we study the design of proportional social welfare functions (SWFs) by adapting tools from approval-based committee voting and participatory budgeting to rank aggregation. In more detail, our central fairness condition is called uPJR and requires that every input ranking with weight α should agree with at least $\lfloor\alpha\binom{m}{2}\rfloor$ pairwise comparisons of the output ranking. We first show that the Squared Kemeny rule, which was suggested by Lederer et al. [31] to proportionally aggregate input rankings, fails even a weakening of this axiom called uJR. We hence design new SWFs and, to this end, prove that uPJR is implied by a more structured fairness notion called rank-priceability. Based on this insight, we design the Proportional Sequential Borda rule (PSB), a remarkable simple rule that satisfies rank-priceability and thus also uPJR. Furthermore, we also prove that PSB guarantees to every subprofile S an average utility that is linear in the size of S, which can be seen as another strong fairness property.

In addition, we suggest two variants of PSB, namely the Ranked Method of Equal Shares (RMES) and the Borda Rule Adjusting the Flow (FB). RMES allows us to connect our approach to the Method of Equal Shares, one of the most prominent tools of fair decision making [36, 37]. Moreover, we demonstrate with this SWF that even rather utilitarian rankings can satisfy uPJR as this rule is guaranteed to assign roughly $\frac{7}{16}$ of the total utility in a utilitarian way. On the other hand, we show that FB satisfies a stronger fairness notion that extends uPJR to arbitary groups of rankings, thus further pushing our understanding of proportionality in rank aggregation.

Our work offers numerous possibilities for future work and we next discuss three particularly interesting directions. *(i)* Given our success in designing SWFs that satisfy variants of PJR in the context of rank aggregation, it seems interesting to analyze stronger fairness notions. One could, for instance, adopt notions such as EJR or core-stability from approval-based committee voting to rank aggregation and aim to find mechanisms satisfying these properties. *(ii)* Interestingly, while most fairness notions in participatory budgeting and committee voting focus on groups of voters with similar preferences, none of our results relies on this idea. Partly, this is because there are multiple ways to define similar input rankings (e.g., we may consider two rankings similar if they have a small swap distance or if they agree on a large prefix) and because it is not clear how to exploit this precondition. However, we would find it interesting to strengthen both our axiomatic and quantitative results by focusing on cohesive groups of rankings. *(iii)* Maybe the biggest restriction of this paper is to define the utility in terms of the pairwise agreement of rankings. While this approach is frequently encountered in the literature, it, e.g., neglects that the first position of the output ranking has often a higher value than other positions. Thus, it seems appealing to extend our results to more general utility functions.

References

[1] Kenneth J. Arrow. 1951. *Social Choice and Individual Values* (1st ed.). New Haven: Cowles Foundation. 2nd edition 1963. [→ p. 3]

[2] Kenneth J. Arrow, Amartya Sen, and Kotaro Suzumura (Eds.). 2002. *Handbook of Social Choice and Welfare*. Vol. 1. North-Holland. [→ p. 2, 4]

[3] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. 2017. Justified Representation in Approval-Based Committee Voting. *Social Choice and Welfare* 48, 2 (2017), 461–485. [→ p. 3, 4, 6]

[4] Haris Aziz, Patrick Lederer, Dominik Peters, Jannik Peters, and Angus Ritossa. 2025. Committee Monotonicity and Proportional Representation for Ranked Preferences. Working paper. [→ p. 4, 5]

[5] Haris Aziz and Nisarg Shah. 2021. Participatory budgeting: Models and approaches. In *Pathways Between Social Science and Computational Social Science: Theories, Methods, and Interpretations*, Tamás Rudas and Gábor Péli (Eds.). Springer International Publishing, 215–236. [→ p. 4]

[6] Niclas Boehmer, Robert Bredereck, and Dominik Peters. 2023. *Rank Aggregation Using Scoring Rules.* Technical Report. https://arxiv.org/pdf/2209.08856. [→ p. 2, 4]

[7] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia (Eds.). 2016. *Handbook of Computational Social Choice.* Cambridge University Press. [→ p. 2, 4]

[8] Markus Brill, Stefan Forster, Martin Lackner, Jan Maly, and Jannik Peters. 2023. Proportionality in Approval-Based Participatory Budgeting. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI).* 5524–5531. [→ p. 3, 4, 8, 9]

[9] Markus Brill and Jonas Israel. 2025. Dynamic proportional rankings. *Social Choice and Welfare* 64 (2025), 221–261. [→ p. 4]

[10] Laurent Bulteau, Noam Hazon, Rutvik Page, Ariel Rosenfeld, and Nimrod Talmon. 2021. Justified Representation for Perpetual Voting. *IEEE Access* 9 (2021), 96598–96612. [→ p. 4]

[11] B. Can and T. Storcken. 2013. Update monotone preference rules. *Mathematical Social Sciences* 65, 2 (2013), 136–149. [→ p. 4]

[12] Diptarka Chakraborty, Syamantak Das, Arindam Khan, and Aditya Subramanian. 2022. Fair Rank Aggregation. In *Advances in Neural Information Processing Systems 35.* 23965–23978. [→ p. 2]

[13] J. R. Chamberlin and P. N. Courant. 1983. Representative Deliberations and Representative Decisions: Proportional Representation and the Borda Rule. *The American Political Science Review* 77, 3 (1983), 718–733. [→ p. 7]

[14] Nikhil Chandak, Shashwat Goel, and Dominik Peters. 2024. Proportional Aggregation of Preferences for Sequential Decision Making. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI).* 9573–9581. [→ p. 4]

[15] Sujoy Chatterjee, Anirban Mukhopadhyay, and Malay Bhattacharyya. 2018. A Weighted Rank aggregation approach towards crowd opinion analysis. *Knowledge-Based Systems* 149 (2018), 47–60. [→ p. 2, 4]

[16] Vincent Conitzer, Matthew Rognlie, and Lirong Xia. 2009. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI).* 109–115. [→ p. 2]

[17] Arthur H. Copeland. 1951. A 'reasonable' social welfare function. Mimeo, University of Michigan Seminar on Applications of Mathematics to the Social Sciences. [→ p. 2]

[18] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar. 2001. Rank aggregation methods for the Web. In *Proceedings of the 10th International Conference on the World Wide Web (WWW).* 613–622. [→ p. 4]

[19] Felix Fischer, Olivier Hudry, and Rolf Niedermeier. 2016. Weighted Tournament Solutions. In *Handbook of Computational Social Choice*, Felix Brandt, Vincent Conitzer, Ulle Endriss, J. Lang, and Ariel D. Procaccia (Eds.). Cambridge University Press, Chapter 4. [→ p. 2]

[20] Rupert Freeman, Markus Brill, and Vincent Conitzer. 2014. On the Axiomatic Characterization of Runoff Voting Rules. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI).* 675–681. [→ p. 4]

[21] Vasilis Gkatzelis, Daniel Halpern, and Nisarg Shah. 2020. Resolving the Optimal Metric Distortion Conjecture. In *Proceedings of the 61st Symposium on Foundations of Computer Science (FOCS).* 1427–1438. [→ p. 16, 36]

[22] John G. Kemeny. 1959. Mathematics without Numbers. *Daedalus* 88 (1959), 577–591. [→ p. 2, 4]

[23] Fatih Erdem Kizilkaya and David Kempe. 2022. Plurality Veto: A simple voting rule achieving optimal metric distortion. In *Proceedings of the 31th International Joint Conference on Artificial Intelligence (IJCAI).* 349–355. [→ p. 16, 36]

[24] Fatih Erdem Kizilkaya and David Kempe. 2023. Generalized Veto Core and a Practical Voting Rule with Optimal Metric Distortion. In *Proceedings of the 24th ACM Conference on Economics and Computation (ACM-EC).* 913–936. [→ p. 16, 36]

[25] R. Kolde, S. Laur, P. Adler, and J. Vilo. 2012. Robust Rank Aggregation for Gene List Integration and Meta-analysis. *Bioinformatics* (2012). [→ p. 2, 4]

[26] Martin Lackner. 2020. Perpetual Voting: Fairness in Long-Term Decision Making. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*. 2103–2110. [→ p. 4]

[27] Martin Lackner and Jan Maly. 2023. Proportional Decisions in Perpetual Voting. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*. 5722–5729. [→ p. 4]

[28] Martin Lackner and Piotr Skowron. 2022. Multi-Winner Voting with Approval Preferences. https://arxiv.org/abs/2007.01795. [→ p. 7]

[29] Martin Lackner and Piotr Skowron. 2023. *Multi-Winner Voting with Approval Preferences*. Springer-Verlag. [→ p. 2, 4]

[30] Patrick Lederer. 2024. Bivariate Scoring Rules: Unifying the Characterizations of Positional Scoring Rules and Kemeny's Rule. *Journal of Economic Theory* 218 (2024), 105836. [→ p. 2]

[31] Patrick Lederer, Dominik Peters, and Tomasz Wąs. 2024. The Squared Kemeny Rule for Averaging Rankings. *Proceedings of the 25th ACM Conference on Economics and Computation (ACM-EC)* (2024). [→ p. 2, 3, 4, 5, 6, 12, 18]

[32] S. Lin. 2010. Rank Aggregation Methods. *WIREs computational statistics* (2010). [→ p. 4]

[33] Tomáš Masařík, Grzegorz Pierczyński, and Piotr Skowron. 2024. A generalised theory of proportionality in collective decision making. In *Proceedings of the 25th ACM Conference on Economics and Computation (ACM-EC)*. 734–754. [→ p. 4]

[34] S. Nitzan and Ariel Rubinstein. 1981. A further characterization of Borda ranking method. *Public Choice* 36, 1 (1981), 153–158. [→ p. 4]

[35] Shuzi Niu, Yanyan Lan, Jiafeng Guo, Xueqi Cheng, Lei Yu, and Cuoping Long. 2015. Listwise approach for rank aggregation in crowdsourcing. In *Proceedings of the 8th ACM International Conference on Web Search and Data Mining*. 253–262. [→ p. 4]

[36] Dominik Peters, Grzegorz Pierczyński, and Piotr Skowron. 2021. Proportional Participatory Budgeting with Additive Utilities. In *Proceedings of the 35th Annual Conference on Neural Information Processing Systems (NeurIPS)*. 12726–12737. [→ p. 3, 4, 13, 18]

[37] Dominik Peters and Piotr Skowron. 2020. Proportionality and the Limits of Welfarism. In *Proceedings of the 21nd ACM Conference on Economics and Computation (ACM-EC)*. 793–794. [→ p. 3, 4, 8, 9, 13, 18]

[38] Evaggelia Pitoura, Kostas Stefanidis, and Georgia Koutrika. 2022. Fairness in Rankings and Recommendations: an overview. *The VLDB Journal* 31, 431–458 (2022). [→ p. 2]

[39] R. C. Prati. 2012. Combining Feature Ranking Algorithms through Rank Aggregation. In *International Conference on Neural Networks*. 1–8. [→ p. 4]

[40] M. E. Renda and U. Straccia. 2003. Web Metasearch: rank vs score based rank aggregation methods. In *ACM Symposium on Applied Computing*. 841–846. [→ p. 4]

[41] Simon Rey and Jan Maly. 2023. *The (Computational) Social Choice Take on Indivisible Participatory Budgeting*. Technical Report. https://arxiv.org/pdf/2303.00621.pdf. [→ p. 2, 4]

[42] Luis Sánchez-Fernández, Edith Elkind, Martin Lackner, N. Fernández, J. A. Fisteus, P. Basanta Val, and Piotr Skowron. 2017. Proportional justified representation. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*. [→ p. 2, 4, 5]

[43] C. Sarkar, S. Cooley, and J. Srivastava. 2014. Robust Feature Selection Technique Using Rank Aggregation. *Applied Artificial Intelligence* 28, 3 (2014), 243–257. [→ p. 2, 4]

[44] Piotr Skowron and Adrian Górecki. 2022. Proportional Public Decisions. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*. 5191–5198. [→ p. 6]

[45] Piotr Skowron, Martin Lackner, Markus Brill, Dominik Peters, and Edith Elkind. 2017. Proportional Rankings. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI)*. 409–415. [→ p. 4]

[46] John H. Smith. 1973. Aggregation of Preferences with Variable Electorate. *Econometrica* 41, 6 (1973), 1027–1041. [→ p. 2, 4]

[47] Siyi Wang, Qi Deng, Shiwei Feng, Hong Zhang, and Chao Ling. 2024. A Survey on Rank Aggregation. In *Proceedings of the 33rd International Joint Conference on Artificial Intelligence (IJCAI)*. 8281–8289. [→ p. 2, 4]

[48] Dong Wei, Md Mouinul Islam, Baruch Schieber, and Senjuti Basu Roj. 2022. Rank aggregation with Proportionate Fairnes. In *Proceedings of the 2022 International Conference on Management of Data (SIGMOD '22)*. 262–275. [→ p. 2]

[49] H. Peyton Young. 1974. A Note on Preference Aggregation. *Econometrica* 42, 6 (1974), 1129–1131. [→ p. 4]

[50] H. Peyton Young. 1988. Condorcet's Theory of Voting. *The American Political Science Review* 82, 4 (1988), 1231–1244. [→ p. 4]

[51] H. Peyton Young and Arthur B. Levenglick. 1978. A Consistent Extension of Condorcet's Election Principle. *SIAM J. Appl. Math.* 35, 2 (1978), 285–300. [→ p. 2, 4]

A Proof of Proposition 1

Proposition 1. *For all $m \geq 5$, there is a profile R and ranking $\succ$ such that $R(\succ) = \frac{m}{5}/\binom{m}{2}$ and $u(\succ, \mathsf{SqK}(R)) = 0$.*

Proof. Fix a number of voters $m \geq 5$ and let $z = \lfloor\frac{(m-1)(m-2)}{4}\rfloor$. We consider the following 4 rankings: $\succ_1 = x_1, x_2 \dots, x_m$, $\succ_2 = x_1, x_m \dots, x_2$, $\succ_3$ is a ranking that puts x_1 last and agrees with exactly d pairs with $\succ_1$, and $\succ_4$ puts x_1 last and orders the remaining pairs exactly inversely to $\succ_3$. Based on these rankings, we will now define a profile R used to prove the proposition. For notational simplicity, we subsequently assume that the weights add up to $\binom{m}{2}$ instead of 1. This is without loss of generality since we can scale down all weights without affecting the outcome. Now, let R be the profile given by

$$R(\succ_1) = \frac{m}{5} \qquad \text{and} \qquad R(\succ_2) = R(\succ_3) = R(\succ_4) = \frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right).$$

Further, let $\rhd$ denote the ranking returned by the Squared Kemeny rule. We will show that $\rhd$ is equal to the ranking $\rhd^* = x_m, \dots, x_1$. This means that the ranking $\succ_1$ is without representation in R, thus proving the proposition. To show this claim, we will introduce additional notation. Specifically, we denote by $\Delta(\succ, \rhd) = |\{(x, y) \in \{C \setminus x_1\}^2 \colon x \succ y \text{ and } y \rhd x\}|$ the swap distance between an arbitrary ranking $\succ$ and $\rhd$ after removing x_1 from both rankings. Furthermore, we define by $d = 1 + |\{x \in C \setminus \{x_1\} \colon x_1 \rhd x\}|$ the position of x_1 in $\rhd$. Given this notation, the swap distance of $\rhd$ to an input rankings $\succ_i$ is $(d-1) + \Delta(\succ_i, \rhd)$ for $i \in \{1, 2\}$ and $(m-d) + \Delta(\succ_i, \rhd)$ for $i \in \{3, 4\}$. We will next show the following auxiliary claims.

(1) It holds that $\Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$.
(2) If $m \in \{5, 6\}$ and $\Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$, then $d > \frac{2m}{3}$.
(3) If $m \geq 7$ and $\Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$, then $d > \frac{2m}{3} + 1$.
(4) If $m \in \{5, 6\}$ and $d > \frac{2m}{3}$, then $\Delta(\rhd, \rhd^*) = 0$.
(5) If If $m \geq 7$ and $d > \frac{2m}{3} + 1$, then $\Delta(\rhd, \rhd^*) = 0$.
(6) If $\Delta(\rhd, \rhd^*) = 0$, then $d = m$.

In combination, these observations clearly imply that $\rhd = \rhd^*$ when $m \geq 5$. We will next prove our auxiliary claims.

Proof of Claim (1): We will show that $\Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$. Assume for contradiction that this is not true and let $\bar{\rhd}$ denote the ranking derived from $\rhd$ by inverting the order over the candidates $\{x_2, \dots, x_m\}$ while keeping the position of x_1 fixed. For instance, if $\rhd = x_2, x_1, x_3, x_4, x_5$, then $\bar{\rhd} = x_5, x_1, x_4, x_3, x_2$. We will show that $\bar{\rhd}$ has a lower cost than $\rhd$ with respect to Squared Kemeny. To this end, we note $\succ_1$ orders the candidates in $\{x_2, \dots, x_m\}$ inversely to $\succ_2$ and that the same holds for $\succ_3$ and $\succ_4$. Hence, we compute that

$$\Delta(\succ_1, \rhd) = \binom{m-1}{2} - \Delta(\succ_2, \rhd) = \Delta(\succ_2, \bar{\rhd}),$$

$$\Delta(\succ_2, \rhd) = \binom{m-1}{2} - \Delta(\succ_1, \rhd) = \Delta(\succ_1, \bar{\rhd}),$$

$$\Delta(\succ_3, \rhd) = \binom{m-1}{2} - \Delta(\succ_4, \rhd) = \Delta(\succ_4, \bar{\rhd}),$$

$$\Delta(\succ_4, \rhd) = \binom{m-1}{2} - \Delta(\succ_3, \rhd) = \Delta(\succ_3, \bar{\rhd}).$$

Now, the cost of $\rhd$ with respect to SqK, denoted by $C_{\mathsf{SqK}}(\rhd)$, is given by

$$\begin{aligned}C_{\mathsf{SqK}}(\rhd) &= R(\succ_1)\cdot(d-1+\Delta(\succ_1,\rhd))^2 + R(\succ_2)\cdot(d-1+\Delta(\succ_2,\rhd))^2\\ &\quad+ R(\succ_3)\cdot(m-d+\Delta(\succ_3,\rhd))^2 + R(\succ_4)\cdot(m-d+\Delta(\succ_4,\rhd))^2.\end{aligned}$$

Furthermore, the cost of $\bar{\rhd}$ is

$$\begin{aligned}C_{\mathsf{SqK}}(\bar{\rhd}) &= R(\succ_1)\cdot(d-1+\Delta(\succ_1,\bar{\rhd}))^2 + R(\succ_2)\cdot(d-1+\Delta(\succ_2,\bar{\rhd}))^2\\ &\quad+ R(\succ_3)\cdot(m-d+\Delta(\succ_3,\bar{\rhd}))^2 + R(\succ_4)\cdot(m-d+\Delta(\succ_4,\bar{\rhd}))^2\\ &= R(\succ_1)\cdot(d-1+\Delta(\succ_2,\rhd))^2 + R(\succ_2)\cdot(d-1+\Delta(\succ_1,\rhd))^2\\ &\quad+ R(\succ_3)\cdot(m-d+\Delta(\succ_4,\rhd))^2 + R(\succ_4)\cdot(m-d+\Delta(\succ_3,\rhd))^2\end{aligned}$$

Next, we observe that $R(\succ_3) = R(\succ_4)$ and $R(\succ_1) < R(\succ_2)$ by the definition of R. Furthermore, we assumed that $\Delta(\rhd,\rhd^*) > \frac{1}{2}\binom{m-1}{2}$, so $\Delta(\succ_1,\rhd) < \frac{1}{2}\binom{m-1}{2} < \Delta(\succ_2,\rhd)$ as $\succ_2$ agrees with $\rhd^*$ on the order of $\{x_2,\dots,x_m\}$ and $\succ_1$ orders these candidates exactly inversely. Using these insights, we obtain that

$$\begin{aligned}&C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\bar{\rhd})\\ &= (R(\succ_1)-R(\succ_2))\,(d-1+\Delta(\succ_1,\rhd))^2 + (R(\succ_2)-R(\succ_1))\,(d-1+\Delta(\succ_2,\rhd))^2\\ &+ (R(\succ_3)-R(\succ_4))\,(m-d+\Delta(\succ_3,\rhd))^2 + (R(\succ_4)-R(\succ_3))\,(m-d+\Delta(\succ_4,\rhd))^2\\ &= (R(\succ_2)-R(\succ_1))\left((d-1+\Delta(\succ_2,\rhd))^2 - (d-1+\Delta(\succ_1,\rhd))^2\right)\\ &> 0.\end{aligned}$$

This shows that $C_{\mathsf{SqK}}(\rhd) > C_{\mathsf{SqK}}(\bar{\rhd})$, which contradicts that $\rhd$ is chosen by the Squared Kemeny rule.

Proofs of Claims (2), (3), and (6): We will next show that x_1 cannot be ranked to highly. To this end, we assume subsequently only that $d \leq m-1$; we will refine this assumption for the proofs of Claims (2) and (3) later on. Now, we first recall that

$$\begin{aligned}C_{\mathsf{SqK}}(\rhd) &= R(\succ_1)\cdot(d-1+\Delta(\succ_1,\rhd))^2 + R(\succ_2)\cdot(d-1+\Delta(\succ_2,\rhd))^2\\ &\quad+ R(\succ_3)\cdot(m-d+\Delta(\succ_4,\rhd))^2 + R(\succ_4)\cdot(m-d+\Delta(\succ_4,\rhd))^2.\end{aligned}$$

Next, let $\rhd'$ denote the ranking derived from $\rhd$ by moving x_1 one position down without reordering any other candidates. This means that x_1 is now the $(d+1)$-th best candidate and that $\Delta(\succ,\rhd') = \Delta(\succ,\rhd)$ for all rankings $\succ \in \mathcal{R}$. Hence, the cost of $\rhd'$ is

$$\begin{aligned}C_{\mathsf{SqK}}(\rhd') &= R(\succ_1)\cdot(d+\Delta(\succ_1,\rhd))^2 + R(\succ_2)\cdot(d+\Delta(\succ_2,\rhd))^2\\ &\quad+ R(\succ_3)\cdot(m-d-1+\Delta(\succ_4,\rhd))^2 + R(\succ_4)\cdot(m-d-1+\Delta(\succ_4,\rhd))^2.\end{aligned}$$

We aim to show that $C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') > 0$. This means that $\rhd$ cannot be chosen by the Squared Kemeny rule as $\rhd'$ has a lower cost. Based on simple calculus, we infer that

$$\begin{aligned}&C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd')\\ &= R(\succ_1)\cdot(-2d-2\Delta(\succ_1,\rhd)+1) + R(\succ_2)\cdot(-2d-2\Delta(\succ_2,\rhd)+1)\\ &+ R(\succ_3)\cdot(2(m-d)+2\Delta(\succ_3,\rhd)-1) + R(\succ_4)\cdot(2(m-d)+2\Delta(\succ_4,\rhd)-1)\end{aligned}$$

Next, we note that $\Delta(\succ_1,\rhd)+\Delta(\succ_2,\rhd) = \binom{m-1}{2}$ and $\Delta(\succ_3,\rhd)+\Delta(\succ_4,\rhd) = \binom{m-1}{2}$, because $\succ_1$ and $\succ_2$ (resp. $\succ_3$ and $\succ_4$) order the candidates in $\{x_2,\dots,x_m\}$ inverse to each other. Furthermore, using the definition of R, we derive that

$$\begin{aligned}
C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') =& \frac{m}{5} \cdot (-2d - 2\Delta(\succ_1, \rhd) + 1) \\
&+ \frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) \cdot (-2d - 2\Delta(\succ_2, \rhd) + 1) \\
&+ \frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) \cdot (2(m-d) + 2\Delta(\succ_3, \rhd) - 1) \\
&+ \frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) \cdot (2(m-d) + 2\Delta(\succ_4, \rhd) - 1) \\
=& \frac{4m}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - 2d \cdot \binom{m}{2} - \frac{1}{3}\binom{m}{2} + \frac{4m}{15} \\
&- 2\left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right)\Delta(\succ_2, \rhd) - \frac{2m}{5}\binom{m-1}{2} \\
&+ 2\left(\frac{1}{3}\binom{m}{2} - \frac{m}{15}\right)\binom{m-1}{2} \\
=& \frac{4m}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - 2d \cdot \binom{m}{2} - \frac{1}{3}\binom{m}{2} + \frac{4m}{15} \\
&+ 2\left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right)\left(\binom{m-1}{2} - \Delta(\succ_2, \rhd)\right)
\end{aligned}$$

In the first equality, we substitute the definition of $R(\succ_i)$ for $i \in \{1, 2, 3, 4\}$. In the next equality, we rearrange the terms: the first line captures all terms that are independent of Δ, the second line uses that $\frac{1}{3}(\binom{m}{2} - \frac{m}{5}) - \frac{m}{5} = \frac{1}{3}\binom{m}{2} - \frac{4m}{15}$ and $\frac{2m}{5}\Delta(\succ_1, \rhd) + \frac{2m}{5}\Delta(\succ_2, \rhd) = \frac{2m}{5}\binom{m-1}{2}$, and the third line applies the same idea for $\Delta(\succ_3, \rhd)$ and $\Delta(\succ_4, \rhd)$. Finally, the last line follows by rearranging our terms.

We now process with a case distinction with respect to m and $\Delta(\rhd, \rhd^*)$ to prove our three claims. To this end, we further note that $\Delta(\succ_2, \rhd^*) = 0$ as $\succ_2$ and $\rhd^*$ agree on the order ov $\{x_2, \dots, x_m\}$. This means that $\Delta(\rhd, \rhd^*) = \Delta(\succ_2, \rhd)$.

Claim (2): We assume that $m \in \{5, 6\}$ and $\Delta(\succ_2, \rhd) = \Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$ and aim to show that $C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') > 0$ if $d \leq \frac{2m}{3}$. To this end, we observe that the assumptions that $\Delta(\succ_2, \rhd) \leq \frac{1}{2}\binom{m-1}{2}$ and $d \leq \frac{2m}{3}$ implies that

$$\begin{aligned}
&C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') \\
&\geq \frac{4m}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - \frac{4m}{3} \cdot \binom{m}{2} - \frac{1}{3}\binom{m}{2} + \frac{4m}{15} + \left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right) \cdot \binom{m-1}{2} \\
&= \left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right) \cdot \binom{m-1}{2} + \frac{4m}{15} - \frac{1}{3}\binom{m}{2} - \frac{4m^2}{15}.
\end{aligned}$$

Since $m \geq 5$, it holds that $\binom{m-1}{2} \geq 6$, so our formula further simplifies to

$$C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') \geq 2\binom{m}{2} - \frac{20m}{15} - \frac{1}{3}\binom{m}{2} - \frac{4m^2}{15}$$

Finally, for $m = 5$, this term evaluates $2\binom{5}{2} - \frac{100}{15} - \frac{1}{3}\binom{m}{2} - \frac{100}{15} = \frac{10}{3}$. Moreover, for $m = 6$, we derive that $2\binom{6}{2} - \frac{120}{15} - \frac{1}{3}\binom{6}{2} - \frac{144}{15} = \frac{375}{15} - \frac{266}{15} > 0$. Hence, in both cases, $\rhd'$ has a lower cost than $\rhd$, contradicting that $\rhd$ is chosen by the Squared Kemeny rule.

Claim (3): Next, we assume that $m \geq 7$ and $\Delta(\succ_2, \rhd) = \Delta(\rhd, \rhd^*) \leq \frac{1}{2}\binom{m-1}{2}$. This time, our goal is to show that $C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') > 0$ if $d \leq \frac{2m}{3} + 1$. Analogous to Claim (2), we derive that

$$\begin{aligned}
&C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd')\\
&\geq \frac{4m}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - \frac{4m}{3}\binom{m}{2} - 2\binom{m}{2} - \frac{1}{3}\binom{m}{2} + \frac{4m}{15} + \left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right)\binom{m-1}{2}\\
&= \left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right) \cdot \binom{m-1}{2} + \frac{4m}{15} - 2\binom{m}{2} - \frac{1}{3}\binom{m}{2} - \frac{4m^2}{15}.
\end{aligned}$$

Furthermore, it holds that $\binom{m-1}{2} \geq 15$ as $m \geq 7$, which implies that

$$\begin{aligned}
C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') &\geq 5\binom{m}{2} - 4m + \frac{4m}{15} - 2\binom{m}{2} - \frac{1}{3}\binom{m}{2} - \frac{4m^2}{15}.\\
&= \frac{8}{3} \cdot \frac{m(m-1)}{2} - 4m + \frac{4m}{15} - \frac{4m^2}{15}\\
&\geq \frac{16}{15}m^2 - 6m\\
&> 0.
\end{aligned}$$

The last inequality here use the fact that $m \geq 7$. This proves again that $C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd')$, so $d > \frac{2m}{3} + 1$ in this case.

Claim (6): Finally, we suppose that $m \geq 5$ is arbitrary and that $\Delta(\rhd, \rhd^*) = 0$. In this case, we will show that $d = m$. To this end, we assume that $d \leq m - 1$ and show that $C_{\mathsf{SqK}}(\rhd) > C_{\mathsf{SqK}}(\rhd')$. Our assumptions imply that

$$\begin{aligned}
&C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd')\\
&\geq \frac{4m}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - 2m\binom{m}{2} + 2\binom{m}{2} - \frac{1}{3}\binom{m}{2} + \frac{4m}{15} + 2\left(\frac{1}{3}\binom{m}{2} - \frac{4m}{15}\right)\binom{m-1}{2}\\
&= \left(\frac{2}{3}\binom{m}{2} - \frac{8m}{15}\right) \cdot \binom{m-1}{2} - \frac{2m}{3}\binom{m}{2} + 2\binom{m}{2} - \frac{4m^2}{15} + \frac{4m}{15} - \frac{1}{3}\binom{m}{2}.
\end{aligned}$$

Now, for $m = 5$, this term evaluates to

$$\left(\frac{2}{3} \cdot 10 - \frac{8 \cdot 5}{15}\right) \cdot 6 - \frac{2 \cdot 5}{3} \cdot 10 + 2 \cdot 10 - \frac{4 \cdot 5^2}{15} + \frac{4 \cdot 5}{15} - \frac{1}{3} \cdot 10 = 44 - \frac{126}{3} = 2.$$

Further for $m = 6$, we get that

$$\left(\frac{2}{3} \cdot 15 - \frac{8 \cdot 6}{15}\right) \cdot 10 - \frac{2 \cdot 6}{3} \cdot 15 + 2 \cdot 15 - \frac{4 \cdot 6^2}{15} + \frac{4 \cdot 6}{15} - \frac{1}{3} \cdot 15 = 98 - 73 = 25.$$

Finally, for $m \geq 7$, we observe that $2\binom{m}{2} - \frac{4m^2}{15} + \frac{4m}{15} - \frac{1}{3}\binom{m}{2} > 0$. Hence, we have that

$$\begin{aligned}
C_{\mathsf{SqK}}(\rhd) - C_{\mathsf{SqK}}(\rhd') &> \left(\frac{2}{3}\binom{m}{2} - \frac{8m}{15}\right) \cdot \binom{m-1}{2} - \frac{2m}{3}\binom{m}{2}\\
&> \frac{2}{3} \cdot \binom{m}{2} \cdot \binom{m-1}{2} - \frac{6m}{5}\binom{m}{2}.
\end{aligned}$$

For the second inequality, we replace the term $-\frac{8}{15}m\binom{m-1}{2}$ with $-\frac{8}{15}m\binom{m}{2}$. Further, it holds that $\frac{2}{3}\binom{m-1}{2} - \frac{6m}{5} > 0$ if $m \geq 7$. Specifically, for $m = 7$, this can be straightforwardly verified and the term is increasing in m when $m \geq 7$. Hence, it holds for all $m \geq 5$ that $C_{\mathsf{SqK}}(\rhd) > C_{\mathsf{SqK}}(\rhd')$ if $d \leq m - 1$ and $\Delta(\rhd, \rhd^*) = 0$, which shows that $d = m$ under these assumptions.

Proofs of Claims (4) and (5): Finally, we will show that, when x_1 is placed low in $\rhd$, then $\Delta(\rhd, \rhd^*) = 0$. To this end, let $\rhd'$ denote the ranking derived from $\rhd$ by ordering all candidates in $\{x_2, \dots, x_m\}$ according to $\rhd^*$ without changing the position of x_1. We aim again to show that $C_{\mathsf{SqK}}(\rhd) > C_{\mathsf{SqK}}(\rhd')$.

We first consider the cost caused by $\succ_3$ and $\succ_4$. Because $\succ_3$ and $\succ_4$ disagree on the order over all candidates, it holds for all rankings $\rhd_1$ with $1 + |\{x \in C \setminus \{x_1\} \colon x \rhd_1 x_1\}| = d$ that the cost caused by $\succ_3$ and $\succ_4$ is

$$R(\succ_3) \cdot (m - d + \Delta(\succ_3, \rhd_1))^2 + R(\succ_4) \cdot (m - d + \binom{m-1}{2} - \Delta(\succ_3, \rhd_1))^2.$$

Using the fact that $R(\succ_3) = R(\succ_4)$, this is equivalent to

$$\begin{aligned}
&R(\succ_3) \cdot \left((m - d + \Delta(\succ_3, \rhd_1))^2 + (m - d + \binom{m-1}{2} - \Delta(\succ_3, \rhd_1))^2\right).\\
=&R(\succ_3) \cdot \Bigg((m-d)^2 + 2(m-d)\Delta(\succ_3, \rhd_1) + \Delta(\succ_3, \rhd_1)^2 + (m-d)^2\\
&+ 2(m-d)(\binom{m-1}{2} - \Delta(\succ_3, \rhd_1)) + (\binom{m-1}{2} - \Delta)^2\Bigg)\\
=&R(\succ_3)\left(2(m-d)^2 + 2(m-d)\binom{m-1}{2} + 2\Delta(\succ_3, \rhd)^2 - 2\Delta(\succ_3, \rhd_1)\binom{m-1}{2} + \binom{m-1}{2}^2\right)
\end{aligned}$$

By considering the first order condition with respect to $\Delta(\succ_3, \rhd_1)$, it is easy to see that this term is minimized when $\Delta(\succ_3, \rhd_1) = \frac{1}{2}\binom{m-1}{2}$. Since quadratic functions grow symmetrically from their minimum, this means that the cost caused by $\succ_3$ and $\succ_4$ is minimal if $\Delta(\succ_3, \Delta_1) = \lfloor \frac{1}{2}\binom{m-1}{2} \rfloor$. We finally note that $\Delta(\succ_3, \rhd') = \Delta(\succ_3, \rhd^*) = \lfloor \frac{1}{2}\binom{m-1}{2} \rfloor$. The first equality here follows because $\rhd'$ and $\rhd$ agree on the order of the candidates $\{x_2, \dots x_m\}$. The second equality holds because $\succ_3$ is chosen such that it agrees with $\succ_1$ on exactly $\lfloor \frac{1}{2}\binom{m-1}{2} \rfloor$ pairs over $\{x_2, \dots, x_m\}$. Since $\rhd^*$ and $\succ_1$ order the candidates in $\{x_2, \dots, x_m\}$ exactly inversely, this shows that $\Delta(\succ_3, \rhd^*) = \lfloor \frac{1}{2}\binom{m-1}{2} \rfloor$. This means that the cost caused by $\succ_3$ and $\succ_4$ is weakly less for $\rhd'$ than for $\rhd$ as $\rhd'$ minimizes the cost for these rankings.

Next, we turn to $\succ_1$ and $\succ_2$. The cost caused by these rankings for a ranking $\rhd_1$ is

$$\begin{aligned}
&R(\succ_1) \cdot \left(d - 1 + \Delta(\succ_1, \rhd_1)\right)^2 + R(\succ_2) \cdot \left(d - 1 + \binom{m-1}{2} - \Delta(\succ_1, \rhd_1)\right)^2\\
=&R(\succ_1) \cdot \left((d-1)^2 + 2(d-1)\Delta(\succ_1, \rhd_1) + \Delta(\succ_1, \rhd_1)^2\right)\\
&+ R(\succ_2) \cdot \Bigg((d-1)^2 + 2(d-1)(\binom{m-1}{2}\\
&- \Delta(\succ_1, \rhd_1)) + \binom{m-1}{2}^2 - 2\binom{m-1}{2}\Delta(\succ_1, \rhd_1) + \Delta(\succ_1, \rhd_1)^2\Bigg).
\end{aligned}$$

We next consider the function f that interprets the above term as a function in $\Delta(\succ_1, \rhd_1)$ and ignores all constant terms. Specifically,

$$f(x) = R(\succ_1) \cdot \left(2(d-1)x + x^2\right) + R(\succ_2) \cdot \left(-2(d-1)x - 2\binom{m-1}{2}x + x^2\right)$$

We next aim to analyze the minimum of $f(x)$, which then gives insight into the optimal swap distance for our above expression. To this end, we first note that the second derivative of f is a positive constant, so the value of f is decreasing until we reach the minimum. Next, we compute the derivative of f:

$$f'(x) = R(\succ_1) \cdot \left(2(d-1) + 2x\right) + R(\succ_2) \cdot \left(-2(d-1) - 2\binom{m-1}{2} + 2x\right).$$

We aim to show that $f'(\binom{m-1}{2}) \leq 0$ as this means that the optimal value of $\Delta(\succ_1, \rhd_1) = \binom{m-1}{2}$. Recall to this end also that $\Delta(\succ_1, \rhd_1) \leq \binom{m-1}{2}$ for every ranking $\rhd_1$. To this end, we observe that $f'(\binom{m-1}{2}) = 2R(\succ_1)\binom{m-1}{2} - 2(R(\succ_2) - R(\succ_1))(d-1)$. We next consider Claims (4) and (5) separately.

Claim (4): First, we assume that $m \in \{5, 6\}$ and $d > \frac{2m}{3}$. Now, if $m = 5$, this means that $d > 10/3$. Furthermore, as d is an integer, we derive that $d \geq 4$. By using the definition of $R(\succ_1)$ and $R(\succ_2)$, we now compute that

$$\begin{aligned}
f'(\binom{m-1}{2}) &= \frac{2m}{5}\binom{m-1}{2} - 2\left(\frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - \frac{m}{5}\right)(d-1) \\
&\leq 2 \cdot 6 - 2 \cdot \left(\frac{1}{3}(10-1) - 1\right) \cdot 3 \\
&= 0
\end{aligned}$$

Similarly, for $m = 6$, the condition that $d > \frac{2m}{3}$ means that $d > 4$. Using again that d is an integer, we get that $d = 5$. Hence, we compute in this case that

$$\begin{aligned}
f'(\binom{m-1}{2}) &= \frac{2m}{5}\binom{m-1}{2} - 2\left(\frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - \frac{m}{5}\right)(d-1) \\
&\leq \frac{12}{5} \cdot 10 - 2 \cdot \left(\frac{1}{3}(15 - \frac{6}{5}) - \frac{6}{5}\right) \cdot 4 \\
&= \frac{-16}{5}
\end{aligned}$$

Hence, in both cases, we get that the minimum of f is reached for $x \geq \binom{m-1}{2}$. Since f differs from the cost of a ranking $\rhd_1$ only in constants, this means that the ranking $\rhd'$ also minimizes the cost caused by $\succ_1$ and $\succ_2$. Put differently, if $m \in \{5, 6\}$ and $d > \frac{2m}{3}$, it holds that $\Delta(\rhd, \rhd_1) = \binom{m-1}{2}$, which equivalently means that $\Delta(\rhd, \rhd^*) = 0$.

Claim (5): As the second case, we suppose that $m \geq 7$ and $d > \frac{2m}{3} + 1$. In this case, we get that

$$\begin{aligned}
f'(\binom{m-1}{2}) &\leq \frac{2m}{5}\binom{m-1}{2} - 2\left(\frac{1}{3}\left(\binom{m}{2} - \frac{m}{5}\right) - \frac{m}{5}\right) \cdot \frac{2m}{3} \\
&= \frac{2m}{5}\binom{m-1}{2} - \frac{4m}{9}\binom{m}{2} + \frac{16m^2}{45} \\
&= \left(\frac{2m}{5} - \frac{4m}{9}\right)\binom{m-1}{2} - \frac{4m(m-1)}{9} + \frac{16m^2}{45} \\
&= -\frac{2}{45}\binom{m-1}{2} - \frac{4m^2}{45} + \frac{4m}{9} \\
&< 0.
\end{aligned}$$

In the last inequality, we use that $\frac{4m^2}{45} > \frac{4m}{9}$ because $m \geq 7$. Hence, we have also in this case that f is minimized for some value $x \geq \binom{m-1}{2}$, so $C_{\mathsf{SqK}}(\rhd) > C_{\mathsf{SqK}}(\rhd')$. This completes the proof of our last auxiliary claim. $\square$

B Proof of Theorem 2

Theorem 2. *Let R be a profile on m candidates and $\rhd = \mathsf{PSB}(R)$ be the ranking chosen by the Proportional Sequential Borda rule. It holds for every subprofile S of R that*

$$\frac{1}{|S|}\sum_{\succ\in\mathcal{R}} S(\succ)u(\succ,\rhd) \geq \binom{m}{2}\cdot\frac{|S|}{4}-\frac{3}{16}.$$

Proof. Fix a profile R and an arbitrary subprofile S of R. Furthermore, let $\rhd = x_1,\dots,x_m$ denote the ranking chosen by PSB and let $b_i(\succ)$ denote the budgets of the input rankings in the i-th round. To simplify the notation, we will assume throughout this proof that $\frac{0}{0}=0$. This assumption removes the need to separately discuss rankings $\succ$ with $R(\succ)=0$, which do not have any influence on PSB. Our proof will focus on the payments made by the rankings in S. We thus define by $b_i^S(\succ)=\frac{S(\succ)}{R(\succ)}b_i(\succ)$ for all rankings $\succ\in\mathcal{R}$ and $i\in\{1,\dots,m\}$ the budget of $b_i(\succ)$ that is due to S. Moreover, we let $c_i^S(\succ)=b_i^S(\succ)-b_{i+1}^S(\succ)=\frac{S(\succ)}{R(\succ)}(b_i(\succ)-b_{i+1}(\succ))$ denote the payment made by $\succ$ in the i-th round with respect to S, and by $C_i^S=\sum_{\succ\in\mathcal{R}}c_i^S(\succ)$ denote the total payment made by the subprofile S in step i.

Now, fix a round $i\in\{1,\dots,m-1\}$ and let $X_i=\{x_i,\dots,x_m\}$. It holds for all ranking $\succ\in\mathcal{R}$ that

$$\begin{aligned} b_i(\succ)-b_{i+1}(\succ) &= \min\left(\frac{(m-i)b_i(\succ)u(\succ,x_i,X_i)}{U(b_i,x_i,X_i)}, b_i(\succ)\right)\\ &\leq \frac{(m-i)b_i(\succ)u(\succ,x_i,X_i)}{U(b_i,x_i,X_i)}. \end{aligned}$$

This means that $c_i^S(\succ)\cdot\frac{U(b_i,x_i,X_i)}{m-i}\leq\frac{S(\succ)}{R(\succ)}\cdot b_i(\succ)\cdot u(\succ,x_i,\dots,X_i)$ for all rankings $\succ\in\mathcal{R}$. Further, we note that $b_i(\succ)\leq b_1(\succ)$ as our budgets are non-increasing and that $b_1(\succ)=R(\succ)\cdot\binom{m}{2}$. Hence, we derive that $c_i^S(\succ)\cdot\frac{U(b_i,x_i,X_i)}{m-i}\leq S(\succ)\cdot\binom{m}{2}\cdot u(\succ,x_i,\{x_i,\dots,x_m\})$. By summing over all rankings, it follows that

$$C_i^S\cdot\frac{U(b_i,x_i,X_i)}{m-i}\leq\sum_{\succ\in\mathcal{R}}S(\succ)\cdot\binom{m}{2}\cdot u(\succ,x_i,X_i).$$

Next, we recall that $\sum_{i=1}^{m-1}(\succ,x_i,\{x_1,\dots,x_m\})=u(\succ,\rhd)$. Hence, we derive that

$$\begin{aligned}\sum_{i=1}^{m-1}C_i^S\cdot\frac{U(b_i,x_i,X_i)}{m-i} &\leq \sum_{i=1}^{m-1}\sum_{\succ\in\mathcal{R}}S(\succ)\cdot\binom{m}{2}\cdot u(\succ,x_i,X_i)\\ &= \sum_{\succ\in\mathcal{R}}S(\succ)\cdot\binom{m}{2}\cdot u(\succ,\rhd)\end{aligned}$$

We will next work towards inferring a lower bound on $U(b_i,x_i,X_i)$. For this, we first recall that $\sum_{\succ\in\mathcal{R}}b_1(\succ)=\frac{m(m-1)}{2}$ by definition. Moreover, it holds that $\frac{m(m-1)}{2}=\sum_{j=1}^{m-1}m-j$. Since we decrease the total budget by at most $m-i$ in each round i, it follows that

$$\sum_{\succ\in\mathcal{R}}b_i(\succ)\geq\sum_{j=1}^{m-1}m-j-\sum_{j=1}^{i-1}m-j=\sum_{j=i}^{m-1}m-j=\frac{(m-i)(m-i+1)}{2}.$$

We next observe that $\sum_{x\in X_i} u(\succ, x, X_i) = \sum_{j=0}^{m-i} j = \frac{(m-i)(m-i+1)}{2}$ for every ranking $\succ \in \mathcal{R}$. Consequently, the Borda score of all candidates in the i-th round is

$$\begin{aligned}\sum_{\succ\in\mathcal{R}}\sum_{x\in X_i} b_i(\succ)u_i(\succ, x, X_i) &= \sum_{\succ\in\mathcal{R}} b_i(\succ)\cdot\frac{(m-i)(m-i+1)}{2}\\ &\geq \frac{(m-i)^2(m-i+1)^2}{4}.\end{aligned}$$

Since there are $m-i+1$ candidates remaining in the i-th round, this means that the average Borda score is at least $\frac{(m-i)^2(m-i+1)}{4}$. We thus infer that $U(b_i, x_i, X_i) \geq \frac{(m-i)^2(m-i+1)}{4}$ because x_i maximizes the Borda score in the i-th round. By substituting this lower bound in our previous inequality, we derive that

$$\sum_{i=1}^{m-1} C_i^S \cdot \frac{(m-i)(m-i+1)}{4} \leq \sum_{\succ\in\mathcal{R}} S(\succ)\cdot\binom{m}{2}\cdot u(\succ, \rhd).$$

Next, we focus on the payments C_i^S. For this, let $C^S = \sum_{i\in S} C_i^S$ denote the total payment made by our subprofile S and let k denote the maximal integer such that $C^S \geq \frac{k(k+1)}{2}$. We note that the term $\frac{(m-i)(m-i+1)}{4}$ is decreasing as i increases, so we minimize the left-hand sum if we pay only in late rounds. Moreover, it holds for all i that $C_i^S \leq m-i$ because the total budget reduction in the i-th step is upper bounded by this value. Since $\frac{k(k+1)}{2} \leq C < \frac{(k+1)(k+2)}{2}$, we thus minimize our sum when $C_i^S = m-i$ for all $i\in\{m-k,\dots,m-1\}$ and $C_{m-k-1}^S = C^S - \frac{k(k+1)}{2} < \frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} = k+1$. For a simple notation, we let $\ell = C^S - \frac{k(k+1)}{2}$ and conclude that

$$\begin{aligned}&\sum_{i=1}^{m-1} C_i^S\cdot\frac{(m-i)(m-i+1)}{4}\\ &\geq \sum_{i=m-k}^{m-1}\frac{(m-i)(m-i)(m-i+1)}{4} + \frac{\ell(m-(m-k-1))(m-(m-k-1)+1)}{4}\\ &= \sum_{i=1}^{k}\frac{i^2(i+1)}{4} + \frac{\ell(k+1)(k+2)}{4}\end{aligned}$$

We will next show that this term is lower bounded by $\frac{C^S(C^S+1)}{4}$. For this, we note that a simple induction shows that $\sum_{i=1}^k i^2(i+1) = \frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6}$. Hence, we have that $\sum_{i=1}^k i^2(i+1) + \ell(k+1)(k+2) = \frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6} + \ell(k+1)(k+2)$. Now, if $k=0$, it holds that $\ell = C^S < 1$. On the other hand, our sum evaluates to $2\ell = 2C^S > C^S(1+C^S)$. Next, suppose that $k\geq 1$. In this case, we observe that

$$\begin{aligned}&\frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6} + \ell(k+1)(k+2)\\ &= \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell(k+1)\right) + \left(\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + \ell(k+1)\right)\end{aligned}$$

Next, we recall that $\ell < k+1$, so $\ell(k+1) > \ell^2$. Further, it is easy to check that $\frac{k^3}{3} + \frac{k}{6} \geq \frac{k^2}{2}$ and that $k^2 \geq \frac{k(k+1)}{2}$. Hence, we further simply our sum to

$$\begin{aligned}
&\frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6} + \ell(k+1)(k+2) \\
&\geq \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell^2\right) + \left(k^2 + \ell(k+1)\right) \\
&\geq \left(\frac{k(k+1)}{2} + \ell\right)^2 + \left(\frac{k(k+1)}{2} + \ell\right) \\
&= C^S(C^S+1)
\end{aligned}$$

This proves our lower bound, so we conclude that $\frac{C^S(C^S+1)}{4} \leq \sum_{\succ \in \mathcal{R}} S(\succ) \cdot \binom{m}{2} \cdot u(\succ, \rhd)$. Finally, we note that proof of Theorem 1 shows that the total remaining budget of PSB is at most $\frac{3}{4}$. In particular, this means that $C^S \geq \sum_{\succ \in \mathcal{R}} S(\succ) \cdot \binom{m}{2} - \frac{3}{4}$. We hence infer that

$$\frac{(\sum_{\succ \in \mathcal{R}} S(\succ) \cdot \binom{m}{2} - \frac{3}{4})(\sum_{\succ S} S(\succ) \cdot \binom{m}{2} + \frac{1}{4})}{4} \leq \sum_{\succ \in \mathcal{R}} S(\succ) \cdot \binom{m}{2} \cdot u(\succ, \rhd).$$

Equivalently, this means that

$$\binom{m}{2} \cdot \frac{\sum_{\succ \in \mathcal{R}} S(\succ)}{4} - \frac{3}{16} \leq \frac{\sum_{\succ \in \mathcal{R}} S(\succ) \cdot \binom{m}{2} \cdot u(\succ, \rhd)}{\sum_{\succ \in \mathcal{R}} S(\succ)\binom{m}{2} + \frac{1}{4}} \leq \frac{\sum_{\succ \in \mathcal{R}} S(\succ) \cdot u(\succ, \rhd)}{\sum_{\succ \in \mathcal{R}} S(\succ)}.$$

Finally, by noting that $\sum_{\succ \in \mathcal{R}} S(\succ) = |S|$, our theorem follows. □

C Proofs Omitted form Section 4.2

Theorem 3. RMES *is well-defined and satisfies rank-priceability.*

Proof. We fix a profile R and show both claims of the theorem independently.

Claim 1: RMES is well-defined.
To show that RMES is well-defined, we need to prove that for each round $i \in \{1, \dots, m-2\}$, there is a candidate x with $\rho_x < \infty$. In particular, note that MES is obviously well-defined in the last round as we simply apply the majority rule for the remaining two candidates. Now, fix some round $i \in \{1, \dots, m-2\}$ and assume that a candidate was bought in all previous rounds. Clearly, if $i = 1$, this assumption is true and it will hold inductively for $i > 1$. Moreover, let X_i denote the candidates that have not been placed in the output ranking yet, $b_i(\succ)$ the remaining budgets, and $b_1(\succ)$ the initial budgets. We first note that for every candidate $x \in X_i$ and ranking $\succ \in \mathcal{R}$, it holds that

$$\min(b_1(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u_i(\succ, x, X_i)) = \min(b_i(\succ), u_i(\succ, x, X_i)).$$

In more detail, if $u(\succ, x, X_i) = 0$, then clearly $\min(b_1(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u_i(\succ, x, X_i)) = u(\succ, x, X_i)$ as all other values are non-negative. On the other hand, if $u(\succ, x, X_i) \geq 1$, then $b_1(\succ)u(\succ, x, X_i) \geq b_i(\succ)$ since $b_1(\succ) \geq b_i(\succ)$. Hence, to show that there is always a candidate that can be bought for a price of $\rho_x < \infty$, it hence suffices to show that there is always a candidate x such that $\sum_{\succ \in \mathcal{R}} \min(b_i(\succ), u(\succ, x, X_i)) \geq m - i$, because this means that x is affordable for a price $\rho \leq 1$.

Now, to prove this claim, we first recall that the total initial weight is $\sum_{\succ \in \mathcal{R}} b_1(\succ) = \binom{m}{2}$ and that we decrease the budget by $m - j$ in all rounds $j < i \leq m-2$. Hence, the total remaining budget at the i-th round is $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$. Next, we proceed with a case distinction and first suppose that there is a ranking $\succ^*$ such that $b_i(\succ^*) \geq m - i - 1$. Furthermore, let x denote the top-ranked candidate of $\succ^*$. If it even holds that $b_i(\succ^*) \geq m-i$, then this ranking alone can afford x by itself because both $b_i(\succ^*) \geq u(\succ^*, x, X_i) = m-i$. On the other hand, if $b_i(\succ^*) < m-i$, then $\min(u(\succ^*, x, X_i), b_i(\succ^*)) = b_i(\succ^*) \geq m-i-1$. Next, let Z denote the set of rankings $\succ$ such that $\succ \neq \succ^*$ and $u(\succ, x, X_i) > 0$ and let $B = \sum_{\succ \in Z} b_i(\succ)$ denote the total remaining budget of these rankings. If $B \geq m-i-b_i(\succ^*)$, x can again be bought. In more detail, if there is a ranking $\succ \in Z$ with $b_i(\succ) \geq 1$, then $\min(b_i(\succ^*), u(\succ^*, x, X_i)) + \min(b_i(\succ), u(\succ, x, X_i)) \geq m-i$. On the other hand, if $b_i(\succ) < 1$ for all $\succ \in Z$, it holds for each of these rankings that $\min(b_i(\succ), u(\succ, x, X_i)) = b_i(\succ)$. Consequently, $\min(b_i(\succ^*), u(\succ^*, x, X_i)) + \sum_{\succ \in Z} \min(b_i(\succ), u(\succ^*, x, X_i)) \geq m-i$. As the last subcase, suppose that $B < m-i-b_i(\succ^*)$ and let y denote the second-best candidate in X_i with respect to $\succ^*$. Since $b_i(\succ^*) \geq m-i-1$ by assumption, it holds that $\min(b_i(\succ^*), u(\succ^*, y, X_i)) = u(\succ^*, y, X_i) = m-i-1$. Furthermore, we observe that all rankings in $\mathcal{R} \setminus (Z \cup \{\succ^*\})$ bottom-rank x. Moreover, we have that

$$
\begin{aligned}
\sum_{\succ \in \mathcal{R} \setminus (Z \cup \{\succ^*\})} b_i(\succ) &= \frac{(m-i)(m-i+1)}{2} - \sum_{\succ \in Z \cup \{\succ^*\}} b_i(\succ) \\
&> \frac{(m-i)(m-i+1)}{2} - (m-i) \\
&\geq 1.
\end{aligned}
$$

Here, the first inequality follows because $B < m-i-b_i(\succ^*)$ and the second one because $m-i \geq 2$. Further, we note that $u(\succ, y, X_i) \geq 1$ for all rankings $\succ \in \mathcal{R} \setminus (Z \cup \{\succ^*\})$ because $y \succ x$. Based on an analogous case distinction as for x when $B \geq m-i-b_i(\succ^*)$, one can now show that y can be afforded.

As the last case, suppose that $b_i(\succ) < m-i-1$ for all $\succ \in \mathcal{R}$. We will consider the total score of the candidates $x \in X_i$. To this end, we will partition the rankings with respect to their remaining budget: B_ℓ denotes the set of rankings $\succ \in \mathcal{R}$ such that $\ell - 1 \leq b_i(\succ) < \ell$. By our assumption that $b_i(\succ) < m-i-1$ for all rankings $\succ$, it holds for each $\succ \in R$ that $\succ \in W_\ell$ for some $\ell \in \{1, \dots m-i-1\}$. Hence, we derive that $\sum_{\succ \in \mathcal{R}} \min(b_i(\succ), u(\succ, x, X_i), w(\succ)) = \sum_{\ell=1}^{m-i-1} \sum_{\succ \in W_\ell} \min(b_i(\succ), u(\succ, x, X_i))$ for all $x \in X_i$. Further, if $\succ \in W_\ell$, then $\min(b_i(\succ), u(\succ, x, X_i)) = b_i(\succ)$ for all $x \in X_i$ with $u(\succ, x, X_i) \geq \ell$. Now, there are $m-i+1$ candidates in X_i as we removed $i-1$ candidates in the previous rounds. Consequently, we have for all $\succ \in W_\ell$ that $\min(b_i(\succ), u(\succ, x, X_i)) = b_i(\succ))$ holds for $m-i+1-\ell$ candidates. Using these insights, we compute that

$$
\begin{aligned}
\sum_{x \in X_i} \sum_{\succ \in \mathcal{R}} \min(b_i(\succ), u(\succ, x, X_i)) &= \sum_{x \in B} \sum_{\ell=1}^{m-i-1} \sum_{\succ \in W_\ell} \min(b_i(\succ), u(\succ, x, X_i)) \\
&\geq \sum_{\ell=1}^{m-i-1} \sum_{\succ \in W_j} (m+1-i-\ell) b_i(\succ) \\
&\geq 2 \sum_{\succ \in \mathcal{R}} b_i(\succ) \\
&= (m-i)(m-i+1).
\end{aligned}
$$

The first and second line here use our previous insights. Next, we use that $(m+1-i-\ell) \geq 2$ since $\ell \leq m-i-1$ and that $\sum_{\ell=1}^{m-i-1} \sum_{\succ \in W_\ell} b_i(\succ) = \sum_{\succ \in \mathcal{R}} b_i(\succ)$. Finally, since the total remaining weight is $\frac{(m-i)(m-i+1)}{2}$, the last step follows. Note that, since there are $m-i+1$ candidates in B, there must be one such that $\sum_{\succ \in \mathcal{R}} \min(b_i(\succ), u(\succ, x, X_i)) \geq m-i$. Hence, ew now conclude that `RMES` is well-defined during the first $m-2$ steps.

Claim 2: RMES satisfies rank-priceability.
Consider the ranking $\rhd = x_1, \dots, x_m$ chosen by RMES for our input profile R. Moreover, let ρ_i denote the price for which candidate x_i is bought for all $i \in \{1, \dots, m-2\}$ and let $b_i(\succ)$ denote the budget of ranking $\succ$ in the i-th step. We will analyze the payments scheme π defined as follows: for $i \in \{1, \dots, m-2\}$, we set $\pi(\succ, x_i) = \min(\rho_i b_1(\succ) u(\succ, x_i, \{x_i, \dots, x_m\}), b_i(\succ), u(\succ, x_i, \{x_i, \dots, x_m\}))$ for all $\succ$. Further, for $i = m-1$, we set $\pi(\succ, x_i) = b_{m-1}(\succ)$ if $x_{m-1} \succ x_m$ and $\pi(\succ, x_i) = 0$ otherwise. Finally, the definition of rank-priceability requires that $\pi(\succ, x_m) = 0$ for all $\succ \in \mathcal{R}$.

We first note that Condition (1) of rank-priceability is satisfies for all $i \in \{1, \dots, m-2\}$ because $\pi(\succ, x_i) \leq u(\succ, x_i, \{x_i, \dots, x_m\})$ by definition of our scheme. Further, when $i = m-1$, then it holds that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = 1$. Since $\pi(\succ, x_{m-1}) = 0$ if $u(\succ, x_{m-1}, \{x_{m-1}, x_m\})$ and $\pi(\succ, x_{m-1}) = b_{m-1}(\succ) \leq 1$ if $u(\succ, x_{m-1}, \{x_{m-1}, x_m\})$, Condition (1) also holds in this round. Secondly, since $\pi(\succ, x_i) \leq b_i(\succ)$ in every step and $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$, Condition (2) of rank-priceability follows. Condition (3) of rank-priceability follows immediately from the definition of RMES because we pay exactly $m - i$ during all rounds $i \in \{1, \dots, m-2\}$ and at most 1 in the $m-1$-th round because the total remaining budget is 1. This also implies Step (4). In more detail, before the $m-1$-th step, the total remaining budget is 1 and we spent at least half on x_i. Hence, the total remaining budget in the end is at most 0.5. Because the total initial budget is $\binom{m}{2}$, this means that the total payments sum up to at least $\binom{m}{2} - 1$. Hence, rank-priceability is indeed satisfied. □

Proposition 4. *Fix a profile R on m candidates and let $\rhd = x_1, \dots, x_m$ denote the ranking chosen by* RMES. *It holds for all $i \in \{1, \dots, \lfloor \frac{m}{4} \rfloor\}$ that $x_i = \arg\max_{x \in \{x_i, \dots, x_m\}} U(b_1, x, \{x_i, \dots, x_m\})$.*

Proof. Fix a profile R, let $\rhd = x_1, \dots, x_m$ denote the ranking chosen by RMES, and let $X_i = \{x_i, \dots, x_m\}$ for all $i \in \{1, \dots, m\}$. We first note that the lemma is trivial for $m \leq 3$ because $\lfloor \frac{m}{4} \rfloor = 0$ in this case. Hence, assume that $m \geq 4$. We will show the lemma by induction and fix a round $i \in \{1, \dots, \lfloor (1 - \frac{m}{4}) m \rfloor\}$. We inductively suppose for all rounds $j \in \{1, \dots, i-1\}$ that $\rho_j = \frac{m-j}{U(b_1, x_j, X_j}$ and $x_j = \arg\max_{x \in X_j} U(b_1, x, X_j)$. Clearly, when $i = 1$, this assumption is true as there were no previous rounds. The central idea of our proof is to show for every input ranking $\succ \in \mathcal{R}$ that

$$\min\left(\frac{(m-i) b_1(\succ) u(\succ, x_i, X_i)}{U(b_1, x_i, X_i)}, b_i(\succ), u(\succ, x_i, X_i)\right) = \frac{(m-i) b_1(\succ) u(\succ, x_i, X_i)}{U(b_1, x_i, X_i)}.$$

This implies that x_i can be bought for a price of $\rho_i = \frac{m-i}{U(b_1, x_i, X_i)}$. Furthermore, if there was an candidate x_k with $U(b_1, x_k, X_i) > U(b_1, x_i, X_i)$, this candidate could be bought for a price of $\frac{m-i}{U(b_1, x_k, X_i)} < \frac{m-i}{U(b_1, x_i, X_i)}$, which contradicts that RMES chooses candidate x_i in the i-th round. So, it follows from our claim also that x_i is the candidate x^* maximizing $U(b_1, x, X_i)$.

To prove the above equality, we fix an input ranking $\succ \in \mathcal{R}$ and let x^* denote the candidate maximizing $U(b_1, x, X_i)$. We will first show that $\frac{m-i}{U(b_1, x^*, X_i)} b_1(\succ) u(\succ, x^*, X_i) \leq u(\succ, x^*, X_i)$. For this, let x denote the top-ranked candidate among X_i with respect to $\succ$. It holds that $u(\succ, x, X_i) = m - i$ as there are $m - i + 1$ candidates remaining. Since x^* maximizes the Borda score, it follows that $U(b_1, x^*, X_i) \geq U(b_1, x, X_i) \geq b_1(\succ) \cdot (m - i)$. Hence, $\frac{m-i}{U(b, x^*, X_i)} b_1(\succ) u(\succ, x^*, X_i) \leq u(\succ, x^*, X_i)$ as required.

Next, we will show that $\frac{m-i}{U(b_1, x^*, X_i)} b_1(\succ) u(\succ, x^*, X_i) \leq b_i(\succ)$. To this end, we observe that, by the induction hypothesis, it holds for all input rankings $\succ \in \mathcal{R}$ that

$$b_i(\succ) = b_1(\succ) - \sum_{j=1}^{i-1} \frac{m-j}{U(b_1, x_j, X_j)} b_1(\succ)(\succ, x_j, X_j).$$

In particular, each ranking $\succ$ pays $\frac{m-j}{U(b_1, x_j, X_j)} b_1(\succ) u(\succ, x_j, X_j)$ in every round $j \in \{1, \dots, i-1\}$. If this was not the case in some round j, candidate x_j could not have been afforded for a price of

$\frac{m-j}{U(b_1,x_j,X_j)}$ as the total payments do not add up to $m-i$. By combining our insights and dividing by $b_1(\succ)$, it suffices to show that

$$\frac{m-i}{U(b_1,x^*,X_i)}u(\succ,x^*,X_i) \leq 1 - \sum_{j=1}^{i-1} \frac{m-j}{U(b_1,x_j,X_j)}u(\succ,x_j,X_j).$$

For this, we observe for every round $k \in \{1,\dots,m-1\}$ that $\sum_{x\in X_k}\sum_{\succ\in\mathcal{R}} b_1(\succ)u(\succ,x,X_k) = \sum_{\succ\in\mathcal{R}} b_1(\succ)\frac{(m-k)(m-k+1)}{2} = \frac{m(m-1)}{2}\cdot\frac{(m-k)(m-k+1)}{2}$. Since x^* maximizes the Borda score in the i-th round, this means that $U(b_1,x^*,X_i) \geq \frac{m(m-1)}{4}\cdot(m-i)$ as the maximum is lower bounded by the average. In turn, it follows that $\frac{m-i}{U(b_1,x^*,X_i} \leq \frac{4}{m(m-1)}$. Analogously, it follows for all candidates x_j with $j \in \{1,\dots,i-1\}$ that $U(b_1,x_j,X_j) \geq \frac{m(m-1)}{4}\cdot(m-j)$. Using these insights, it follows that

$$\frac{m-i}{U(b_1,x^*,X_i)}u(\succ,x^*,X_i) \leq \frac{4}{m(m-1)}u(\succ,x^*,X_i) \qquad \text{and}$$

$$1-\sum_{j=1}^{i-1}\frac{m-j}{U(b_1,x_j,X_j)}u(\succ,x_j,X_j) \geq 1-\sum_{j=1}^{i-1}\frac{4}{m(m-1)}u(\succ,x_j,X_j).$$

We will show that $\frac{4}{m(m-1)}u(\succ,x^*,X_i) \leq 1-\sum_{j=1}^{i-1}\frac{4}{m(m-1)}u(\succ,x_j,X_j)$. Equivalently, we can prove that $u(\succ,x^*,X_i)+\sum_{j=1}^{i-1}u(\succ,x_j,X_j) \leq \frac{m(m-1)}{4}$. To this end, we note that $u(\succ,x^*,X_i)\leq m-i$ and $u(\succ,x_j,X_j) \leq m-j$ for all $j\in\{1,\dots,m-1\}$. Hence, this inequality is satisfied when $\sum_{j=1}^{i} m-j \leq \frac{m(m-1)}{4}$. Next, we compute that $\sum_{j=1}^{i} m-j = \sum_{j=1}^{m-1} j - \sum_{j=1}^{m-i-1} i = \frac{m(m-1)}{2} - \frac{(m-i)(m-i-1)}{2}$. Consequently, we can further reduce our problem to showing that

$$\frac{m(m-1)}{2}-\frac{(m-i)(m-i-1)}{2} \leq \frac{m(m-1)}{4}.$$

Finally, we will use that $i \leq \frac{m}{4}$ and that $m \geq 4$. In particular, this implies that

$$\frac{(m-i)(m-i-1)}{2} \geq \frac{\frac{3m}{4}(\frac{3m}{4}-1)}{2} = \frac{9m^2}{32}-\frac{3m}{8} = \frac{m(m-1)}{4}+\frac{m^2}{32}-\frac{m}{8} \geq \frac{m(m-1)}{4}.$$

Hence, it indeed holds that $\frac{m(m-1)}{2}-\frac{(m-i)(m-i-1)}{2} \leq \frac{m(m-1)}{4}$. ◻

Theorem 4. *Let R be a profile on $m \geq 4$ candidates, $\rhd = \mathsf{RMES}(R)$, and define $\xi = \binom{m-\lfloor\frac{m}{4}\rfloor}{2}$. It holds for every subprofile S of R that*

$$\frac{1}{|S|}\sum_{\succ\in\mathcal{R}} S(\succ)u(\succ,\rhd) \geq \begin{cases} \binom{m}{2}\cdot\frac{|S|}{4}-\frac{1}{8} & \text{if } \binom{m}{2}|S|-0.5\leq\xi \\ \frac{1}{2}\cdot\binom{m}{2}\cdot(1-\frac{\xi}{\binom{m}{2}|S|})+\frac{\xi+1}{4}\cdot\frac{\xi}{\binom{m}{2}|S|}-\frac{1}{4|S|} & \text{if } \binom{m}{2}|S|-0.5>\xi. \end{cases}$$

Proof. Let R and S be defined as in the theorem and let $\rhd = x_1,\dots,x_m$ denote a ranking chosen by RMES for R. As usual, we let $X_i = \{x_i,\dots,x_m\}$. By the definition of RMES, there is for every candidates $x_i \in \{x_1,\dots,x_{m-2}\}$ a price ρ_i for which it is bought and we define $\rho_{m-1}=1$ for notational ease. Moreover, let $b_i(\succ)$ denote the budget of every input ranking $\succ\in\mathcal{R}$ in the i-th round. In particular, $b_1(\succ)=R(\succ)\cdot\binom{m}{2}$ for all $\succ\in\mathcal{R}$. Next, we let $b_i^S(\succ)=\frac{S(\succ)}{R(\succ)}b_i(\succ)$ for all $\succ\in\mathcal{R}$. As in Theorem 2, we assume here (and henceforth) that $\frac{0}{0}=0$ for the sake of simple notation. Finally, we let $c_i^S(\succ)=b_i^S(\succ)-b_{i+1}^S(\succ)=\frac{S(\succ)}{R(\succ)}\cdot\min(\rho_i b_1(\succ)u(\succ,x_i,X_i),b_i(\succ),u(\succ,x_i,X_i))$ be the payment of a single ranking $\succ$ and $C_i^S=\sum_{\succ\in\mathcal{R}}\frac{S(\succ)}{R(\succ)}C_i^S(\succ)$.

Now, analogous to the proof of Theorem 6, it can be shown that

$$\sum_{i=1}^{m-1} \frac{C_i^S}{\rho_i} \leq \sum_{i=1}^{m-1} \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, x_i, X_i) = \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, \rhd).$$

We will next work to derive upper bounds for all prices ρ_i since this allows us to lower bound our left hand sum. First, it holds that $\rho_{m-1} = 1$, so we are done with this case. Similar, we will use that $\rho_{m-2} \leq 1$ as we have shown in the proof of ?? that there is a candidate that is affordable for this price. Secondly, Proposition 4 shows that $\rho_i = \frac{m-i}{U(b_1, x_i, X_i)}$ for all $i \in \{1, \dots, \lfloor \frac{m}{4} \rfloor\}$. Moreover, we have shown in the proof of this proposition that $U(b_i, x_i, X_i) \geq \frac{m(m-1)(m-i)}{4}$, so $\rho_i \leq \frac{4}{m(m-1)}$ for these rounds. Finally, for $i \in \{\lfloor \frac{m}{4} + 1 \rfloor, \dots, m-3\}$, we note that $b_i(\succ) \leq b_1(\succ)$ for all $\succ$. This implies for all candidate $x \in X_i$ and all values of $\rho \geq 0$ that

$$\begin{aligned}&\sum_{\succ \in \mathcal{R}} \min(\rho \cdot b_1(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u(\succ, x_j, X_i)) \\ \geq &\sum_{\succ \in \mathcal{R}} \min(\rho \cdot b_i(\succ) \cdot u(\succ, x_j, X_i), b_i(\succ), u(\succ, x_j, X_i)).\end{aligned}$$

Now, let ρ and ρ' denote the minimal values for which there are candidates $x, x' \in X_i$ such that $\sum_{\succ \in \mathcal{R}} \min(\rho \cdot b_1(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u(\succ, x, X_i)) = m - i$ and $\sum_{\succ \in \mathcal{R}} \min(\rho' \cdot b_i(\succ) \cdot u(\succ, x_j, X_i), b_i(\succ), u(\succ, x_j, X_i)) = m - i$. By our previous observation, we derive that $\rho \leq \rho'$. We will next show that there is a candidate $x \in X_i$ such that $\sum_{\succ \in \mathcal{R}} \min(\frac{4}{(m-i+1)(m-i)} \cdot b_i(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u(\succ, x_j, X_i)) \geq m - i$, thus proving that $\rho \leq \rho' \leq \frac{4}{(m-i+1)(m-i)}$.

For this, we first observe that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i+1)(m-i)}{2}$ as the initial total budget is $\frac{m(m-1)}{2}$ and we pay $m - j$ in each round $j \in \{1, \dots, i-1\}$. Now, if there is a ranking $\succ$ such that $b_i(\succ) \geq \frac{(m-i+1)(m-i)}{4}$, we can choose x as the top-ranked candidate in $\succ$. Since there are $m - i + 1$ candidates remaining, it holds that $u(\succ, x, X_i) = m - i$. Furthermore, we have that $b^i(\succ) \geq \frac{(m-i+1)(m-i)}{4} \geq \frac{4(m-i)}{4} = m - i$ as $i \leq m - 3$. Finally, we observe that $\frac{4}{(m-i+1)(m-i)} \cdot b_i(\succ) \cdot u(\succ, x, X_i) \geq u(\succ, x, X_i) = m - i$. Hence, the ranking $\succ$ itself is able to buy x for the price of $\frac{4}{(m-i+1)(m-i)}$.

As the second case, suppose that $b_i(\succ) < \frac{(m-i+1)(m-i)}{4}$ for all $\succ \in \mathcal{R}$. In this case, we first note that $\frac{4}{(m-i+1)(m-i)} b_i(\succ) u(\succ, x, X_i) < u(\succ, x, X_i)$ for all $\succ \in \mathcal{R}$ and $x \in X_i$ as $\frac{4}{(m-i+1)(m-i)} \cdot b_i(\succ) < 1$. Moreover, because $m - i + 1 \geq 4$ and $u(\succ, x, X_i) \leq m - i$, it holds that $\frac{4}{(m-i+1)(m-i)} \cdot u(\succ, x_j, X_i) \leq 1$, so $\frac{4}{(m-i+1)(m-i)} \cdot b^i(\succ) u(\succ, x_j, X_i) \leq b^i(\succ)$. We now conclude that $\min(\frac{4}{(m-i+1)(m-i)} \cdot b_i(\succ) \cdot u(\succ, x, X_i), b_i(\succ), u(\succ, x, X_i)) = \frac{4}{(m-i+1)(m-i)} \cdot b_i(\succ) \cdot u(\succ, x_j, X_i)$ for all candidates $x \in X_i$ and rankings $\succ \in \mathcal{R}$. Next, we compute that

$$\begin{aligned}&\frac{1}{m-i+1} \sum_{x \in X_i} \sum_{\succ \in \mathcal{R}} \min\left(\frac{4}{(m-i+1)(m-i)} b_i(\succ) u(\succ, x, X_i), b_i(\succ), u(\succ, x, X_i)\right) \\ =&\frac{1}{m-i+1} \sum_{\succ \in \mathcal{R}} \sum_{x \in X_i} \frac{4}{(m-i+1)(m-i)} b_i(\succ) u(\succ, x_j, X_i) \\ =&\frac{4}{(m-i)(m-i+1)^2} \sum_{\succ \in \mathcal{R}} b_i(\succ) \sum_{j=0}^{m-i} j \\ =&\frac{2}{(m-i+1)} \sum_{\succ \in \mathcal{R}} b_i(\succ) \\ =&m - i\end{aligned}$$

This means that at least one candidate is affordable for a price of $\frac{4}{(m-i+1)(m-i)}$, so $\rho \leq \rho' \leq \frac{4}{(m-i)(m-i+1)}$ holds also in this case.

Using our bounds on ρ_i, we conclude that

$$\sum_{i=1}^{\lfloor \frac{m}{4} \rfloor} \frac{C_i^S m(m-1)}{4} + \sum_{i=\lfloor \frac{m}{4} \rfloor+1}^{m-3} \frac{C_i^S (m-i+1)(m-i)}{4} + C_{m-2}^S + C_{m-1}^S \leq \sum_{\succ \in \mathcal{R}} b_S(\succ) u(\succ, \rhd).$$

Next, let $C^S = \sum_{i=1}^{m-1} C_i^S$ denote the total payments made by S. We note that $C^S \geq \sum_{\succ \in \mathcal{R}} b_1^S(\succ) - 0.5$ as the total remaining budget of RMES is at most 0.5. Furthermore, let k denote the largest integer such that $C^S \geq \frac{k(k+1)}{2}$. We proceed with a case distinction regarding C^S and first assume that $C \leq 3$. In this case, we minimize the left-hand side of our inequality if $c_i = 0$ for all $i \in \{1, \dots, m-3\}$. Hence, we derive that $\sum_{\succ \in \mathcal{R}} b_1^S(\succ) - 0.5 \leq C^S \leq \sum_{\succ \in \mathcal{R}} b_S(\succ) u(\succ, \rhd)$. By dividing by $\sum_{\succ \in \mathcal{R}} b_1^S(\succ) = |S|\binom{m}{2}$, we derive that

$$1 - \frac{1}{m(m-1)|S|} \leq \frac{1}{|S|} \cdot \sum_{\succ \in \mathcal{R}} S(\succ) \cdot u(\succ, \rhd)$$

Now, to prove our theorem in this case, we first note that $\sum_{\succ \in \mathcal{R}} S(\succ) \cdot u(\succ, \rhd) \geq \binom{m}{2} \cdot \frac{|S|}{4} - \frac{1}{8}$ holds trivially if $|S| \leq \frac{1}{2}\binom{m}{2}^{-1}$ because $\frac{|S|}{4} - \frac{1}{8} \leq 0$ int his case. Next, we note that $C \leq 3$ implies that $|S| \leq \frac{7}{2}\binom{m}{2}^{-1}$. We can thus prove our theorem in this case by showing that $\binom{m}{2} \cdot \frac{|S|}{4} - \frac{1}{8} \leq 1 - \frac{1}{m(m-1)|S|}$. Equivalently, we can show that $2\binom{m}{2}^2 |S|^2 - 9\binom{m}{2}|S| + 4 \leq 0$. Since the left-hand term is a quadratic function, it suffices to note that $2\binom{m}{2}^2|S|^2 - 9\binom{m}{2}|S| + 4 = 0$ if $|S| = 4\binom{m}{2}^{-1}$ and $|S| = \frac{1}{2}\binom{m}{2}^{-1}$. This proves our inequality, so the theorem holds in this case.

Next, suppose that $3 \leq C \leq \xi$. In this case, we minimize $\sum_{i=1}^{\lfloor \frac{m}{4} \rfloor} \frac{C_i^S m(m-1)}{4} + \sum_{i=\lfloor \frac{m}{4} \rfloor+1}^{m-3} \frac{C_i^S (m-i+1)(m-i)}{4} + C_{m-2}^S + C_{m-1}^S$ if the group S only pays for candidates chosen in late rounds. However, it holds that $C_i^S \leq m - i$ as the i-th candidate has a cost of $m - i$. Hence, we minimize our sum by setting $C_i^S = m - i$ for all $i \in \{m-k, \dots, m-1\}$ and $C_{m-k-1}^S = C - \frac{k(k+1)}{2}$. Since $C_{m-1}^S = 1$ and $C_{m-2}^S = 2$, so it holds that $C_{m-2}^S + C_{m-1}^S = \frac{3}{2}C_{m-2}^S + \frac{1}{2}C_{m-1}^S - \frac{1}{2} = \frac{3 \cdot 2}{4}c_{m-2} + \frac{2 \cdot 1}{4}c_{m-1} - \frac{1}{2}$. Further, by the assumption that $C \leq \epsilon = \frac{(m-\lfloor \frac{m}{4} \rfloor)(m-\lfloor \frac{m}{4} \rfloor-1)}{2} = \sum_{j=1}^{m-\lfloor \frac{m}{4} \rfloor-1} j$, we derive that $k \leq m - \lfloor \frac{m}{4} \rfloor - 1$ and this inequality is strict unless $C^S = \xi$. Hence, it holds that $C_i^S = 0$ for all $i \in \{1, \dots, \lfloor \frac{m}{4} \rfloor\}$. Next, let $\ell = C^S - \frac{k(k+1)}{2}$. By our reasoning so far, we have that

$$\sum_{i=m-k}^{m-1} \frac{(m-i)(m-i)(m-i+1)}{4} + \frac{\ell(k+1)(k+2)}{4} - \frac{1}{2} \leq \sum_{\succ \in \mathcal{R}} b_S(\succ) u(\succ, \rhd).$$

Next, we note analogous to the proof of Theorem 2 that

$$\begin{aligned}
&\sum_{i=m-k}^{m-1} (m-i)(m-i)(m-i+1) + \ell(k+1)(k+2) \\
&= \sum_{i=1}^{k} i^2(i+1) + \ell(k+1)(k+2) \\
&= \frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6} + \ell(k+1)(k+2) \\
&= \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell(k+1)\right) + \left(\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + \ell(k+1)\right)
\end{aligned}$$

We observe that $\ell \leq k+1$, so $\ell(k+1) \geq \ell^2$. Furthermore, since $k \geq 2$, it holds that $\frac{k^3}{3} + \frac{k}{6} \geq \frac{k^2}{2} + 1$ and $k^2 \geq \frac{k(k+1)}{2}$. Hence, we derive that

$$\begin{aligned}\sum_{i=1}^{k} i^2(i+1) + \ell(k+1)(k+2) &\geq \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell^2\right) + \left(k^2 + 1 + \ell(k+1)\right) \\ &\geq \left(\frac{k(k+1)}{2} + \ell\right)^2 + \left(\frac{k(k+1)}{2} + \ell\right) + 2 \\ &= C^S(C^S+1) + 2.\end{aligned}$$

Substituting this into our original inequality shows that $\frac{C(C+1)}{4} \leq \sum_{\succ \in \mathcal{R}} b_i^S(\succ) u(\succ, \rhd)$. Finally, from here on, we can complete the proof analogously to the proof of Theorem 2.

As last case suppose that $C > \xi$. In this case, we minimize the sum $\sum_{i=1}^{\lfloor \frac{m}{4} \rfloor} \frac{C_i^S m(m-1)}{4} + \sum_{i=\lfloor \frac{m}{4} \rfloor+1}^{m-3} \frac{C_i^S(m-i+1)(m-i)}{4} + C_{m-2}^S + C_{m-1}^S$ by setting $C_i^S = m - i$ for all $i \in \{\lfloor \frac{m}{4} + 1 \rfloor, \dots, m-1\}$ and distributing the remaining budget arbitrarily on the rounds $i \in \{1, \dots, \lfloor \frac{m}{4} \rfloor\}$. We note that this means that we spend a budget of ξ in the rounds $\{\lfloor \frac{m}{4} + 1 \rfloor, \dots, m-1\}$. Hence, we can use the computations from the previous case to derive that $1 + 2 + \sum_{i=m-\lfloor \frac{m}{4} \rfloor - 1}^{m-i} \frac{(m-i)(m-i)(m-i+1)}{4} \geq \frac{\xi(\xi+1)}{4}$. This follows effectively by assuming $C^S = \xi$ and repeating our argument. On the other hand, in the rounds $i \in \{1, \dots, \lfloor \frac{m}{4} \}$, we obtain a utility of $\frac{m(m-1)}{4} \sum_{i=1}^{\lfloor \frac{m}{4} \rfloor} C_i^S = \frac{m(m-1)}{4}(C^S - \xi)$. In summary, we conclude that

$$\frac{m(m-1)}{4}(C^S - \xi) + \frac{\xi(\xi+1)}{2} \leq \sum_{\succ \in \mathcal{R}} b_S(\succ) u(\succ, \rhd).$$

Next, we use that $C^s \geq \binom{m}{2}|S| - 0.5$ and divide by $\binom{m}{2}|S|$ to derive that

$$\frac{1}{2} \cdot \binom{m}{2} \left(1 - \frac{1}{2\binom{m}{2}|S|} - \frac{\xi}{\binom{m}{2}|S|}\right) + \frac{\xi(\xi+1)}{2\binom{m}{2}|S|} \leq \frac{1}{|S|} \sum_{\succ \in \mathcal{R}} S(\succ) u(\succ, \rhd).$$

Since this expression is equivalent to the term in the theorem, this completes the proof. □

D Proofs Omitted from Section 4.3

Proposition 5. *The following claims are true:*

(1) If a ranking is pair-priceable for a profile, it also satisfies sPJR.
(2) For every profile, there is a pair-priceable ranking.

Proof. We will show both claims independently from each other.

Claim (1): Pair-priceability implies sPJR. Fix a profile R on m candidates and suppose that $\rhd$ is a pair-priceable ranking for R. Moreover, we denote by S an arbitrary subprofile of R and aim to show that $|A(\rhd) \cap \bigcup_{\succ \in \mathcal{R}\colon S(\succ)>0} A(\succ)| \geq \lfloor S(\succ) \cdot \binom{m}{2} \rfloor$. To this end, let π denote the payment scheme that verifies the pair-priceability of $\rhd$. By Condition (4) of pair-priceability, we have that $\sum_{\succ \in \mathcal{R}} \sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) > \binom{m}{2} - 1$. Since the total budget of all rankings is $\binom{m}{2}$, this implies that

$$\sum_{\succ \in \mathcal{R}\colon S(\succ)>0} \sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) > \binom{m}{2} \cdot \sum_{\succ \in \mathcal{R}\colon S(\succ)>0} R(\succ) - 1 \geq |S| \cdot \binom{m}{2} - 1.$$

Next, by Conditions (1) and (3), each ranking $\succ$ only pays for pairs $(x_i, x_j) \in A(\succ)$ and we can pay at most 1 to each such pair. Put differently, this means that the rankings with positive weight in

S can only pay for the pairs in $A(\rhd) \cap \bigcup_{\succ \in \mathcal{R}\colon S(\succ)>0} A(\succ)$ and at most 1 for each such pair. Hence, it holds that

$$\sum_{\succ \in \mathcal{R}\colon S(\succ)>0} \sum_{(x_i,x_j)\in A(\rhd)} \pi(\succ, (x_i, x_j)) \leq |A(\rhd) \cap \bigcup_{\succ \in \mathcal{R}\colon S(\succ)>0} A(\succ)|.$$

By combining our two inequalities, we conclude that

$$|S| \cdot \binom{m}{2} - 1 < |A(\rhd) \cap \bigcup_{\succ \in \mathcal{R}\colon S(\succ)>0} A(\succ)|.$$

Finally, since the right side of this inequality is an integer, it follows that $\lfloor |S| \cdot \binom{m}{2} \rfloor \leq |A(\rhd) \cap \bigcup_{\succ \in \mathcal{R}\colon S(\succ)>0} A(\succ)|$, thus proving that sPJR holds.

Claim (2): Pair-priceable rankings are guaranteed to exist Fix a profile R on m candidates. We will construct a pair-priceable ranking in rounds and, in every round, we will identify a candidate x_i such that the cost of the pairs (x_i, x) can be fully covered for all remaining candidates x. To make this more formal, fix some round $i \in \{1, \dots, m-2\}$ and let $b_i(\succ)$ denote the remaining budgets of the rankings and X_i the remaining candidates. In the first round, we have that $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ and $X_1 = C$. We further assume that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$, which is true for the first round by the definition of b_1. Now, in the i-th round, we aim to find a candidate x^* and a payment function $\pi_i : \mathcal{R} \times X_i \setminus \{x^*\} \to [0, 1]$ such that *(i)* $\sum_{x\in X_i \setminus \{x^*\}} \pi_i(\succ, x) \leq b_i(\succ)$ for all $\succ \in \mathcal{R}$, *(ii)* $\pi_i(\succ, x) = 0$ for all $\succ \in \mathcal{R}$ and $x \in X_i$ with $x \succ x^*$, and *(iii)* $\sum_{\succ \in \mathcal{R}} \pi_i(\succ, x) = 1$ for all $x \in X_i$. We note that the payment scheme π_i can also be interpreted as a payment scheme for the pairs (x^*, x).

For identifying such a candidate x^* and the payment scheme π_i, we will introduce domination graphs, which are studied in the context of metric distortion of voting rules [e.g., 21, 23, 24]. Specifically, the domination graph of a candidate x is given by $G_x = (\mathcal{R}, X_i, E_x)$, where $(\succ, y) \in E_x$ if and only if $x \succ y$ or $x = y$. Less formally, in the domination graph of x, every ranking $\succ$ has an edge to every candidate $y \in X_i \setminus \{x\}$ that is weakly worse than x in $\succ$. Our interest in these domination graphs comes from the ranking-matching lemma shown in the previously cited papers: for any two functions $p : \mathcal{R} \to \mathbb{R}_{>=0}$ and $q : X_i \to \mathbb{R}_{>=0}$ such that $\sum_{\succ \in \mathcal{R}} p(\succ) = \sum_{x\in X_i} q(x) > 0$, there is a candidate x^* and a matching $\mu : \mathcal{R} \times C \to \mathbb{R}_{\geq 0}$ in the domination graph G_x such that *(a)* $p(\succ) = \sum_{x\in X_i} \mu(\succ, x)$ for all $\succ \in \mathcal{R}$, *(b)* $q(x) = \sum_{\succ \in \mathcal{R}} \mu(\succ, x)$ for all $\succ \in \mathcal{R}$, and *(c)* $\mu(\succ, x) = 0$ for all $(\succ, x) \notin E_{x^*}$.

Next, let q denote a weight function over the candidates X_i given by $q(x) = \frac{m-i}{2}$. We note that $q(x) \geq 1$ because we assume that $i \leq m-2$, and that $\sum_{x\in X_i} q(x) = \frac{(m-i)(m-i+1)}{2} = \sum_{\succ \in \mathcal{R}} b_i(\succ)$ because X_i contains $m-i+1$ candidates. Hence, the ranking-matching lemma shows that there is a candidate x^* and a matching μ that satisfies the Conditions *(a)*, *(b)*, and *(c)*. Based on μ, we define the payment scheme π_i for x^* by $\pi_i(\succ, x) = \frac{2}{m-i}\mu(\succ, x)$ for all $\succ \in \mathcal{A}$ and $x \in X_i \setminus \{x^*\}$. We next show that π_i satisfies Conditions *(i)*, *(ii)*, and *(iii)* (of the first paragraph). In more detail, Condition *(i)* holds because $\sum_{x\in X_i\setminus\{x^*\}} \pi_i(\succ, x) = \sum_{x\in X_i\setminus\{x^*\}} \frac{2}{m-i}\mu(\succ, x) \leq \sum_{x\in X_i} \mu(\succ, x) = b_i(x)$ for every ranking $\succ$, where the last equality uses Condition *(a)* of μ. Further, Condition *(ii)* is true because $(\succ, x) \notin E_{x^*}$ if $x \succ x^*$ and thus $\pi_i(\succ, x) = \frac{2}{m-i}\mu(\succ, x) = 0$ by Condition *(c)* of μ. Finally, condition *(iii)* follows since $\sum_{\succ \in \mathcal{R}} \pi_i(\succ, x) = \sum_{\succ \in \mathcal{R}} \frac{2}{m-i}\mu(\succ, x) = 1$ for all $x \in X_i \setminus \{x^*\}$, where we use Condition *(b)* of μ in the last step.

After identifying the candidate x^* and its payment function π_i, we place x^* at the i-th position of our output ranking, set $X_{i+1} = X_i \setminus \{x^*\}$ and $b_{i+1}(\succ) = b_i(\succ) - \sum_{x\in X_i\setminus\{x^*\}} \pi(\succ, x)$ for all $\succ \in \mathcal{R}$, and proceed with the next round. Since we deduct a total budget by $m-i$ in this round, the remaining budget in the next round will be $\frac{(m-i)(m-i+1)}{2} - (m-i) = \frac{(m-i-1)(m-i)}{2}$, so this precondition of our construction remains true.

We observe that the above scheme only works when $i \leq m-2$, which leaves open what to do in the last round. In this case, we will use a simple majority vote over the last two candidates x, y with respect to the remaining budgets to determine the winner: if $\sum_{\succ \in \mathcal{R}\colon x \succ y} b_{m-1}(\succ) > \sum_{\succ \in \mathcal{R}\colon y \succ x} b_{m-1}(\succ)$, we place x at position $m-1$ of the output ranking and y at the last position. On the other hand, if $\sum_{\succ \in \mathcal{R}\colon x \succ y} b_{m-1}(\succ) < \sum_{\succ \in \mathcal{R}\colon y \succ x} b_{m-1}(\succ)$, we place y at position $m-1$ and x at position m. A majority tie can be resolved arbitrarily. Furthermore, denote by x^* the candidate that is ranked at position $m-i$ and by y the candidate that is ranked last. We define the payment scheme π_{m-1} of this step by $\pi_{m-1}(\succ, y) = b_i(\succ)$ if $x^* \succ y$ and $\pi_{m-1}(\succ, y) = 0$ if $y \succ x^*$. We note here also that $b_{m-1}(\succ) \leq 1$ for all $\succ \in \mathcal{R}$ because $\sum_{\succ \in \mathcal{R}} b_{m-1}(\succ) = \frac{(m-(m-1))(m-(m-1)+1)}{2} = 1$.

Finally, let $\rhd = x_1, \dots, x_m$ denote the ranking constructed by the above process and let π_i denote the payment scheme for every step. We define the global payment scheme $\pi : \mathcal{R} \times A(\rhd) \to [0,1]$ by $\pi(\succ, (x_i, x_j)) = \pi_i(\succ, x_j)$ for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in A(\rhd)$. We claim that π satisfies all conditions of rank-priceability. For Condition (1), we note for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in A(\rhd)$ that $\pi(\succ, (x_i, x_j)) = \pi_i(\succ, x_j) = 0$ if $x_j \succ x_i$ and $\pi(\succ, (x_i, x_j)) \leq 1$ if $x_j \succ x_i$ since $\sum_{\succ \in \mathcal{R}} \pi_i(\succ, x_j) = 1$. Condition (2) follows because $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ for all $\succ \in \mathcal{R}$ and we never increase the budget of a ranking in our process. Moreover, our budgets are never negative as $\pi(\succ, x_i)$ is always upper bounded by $b_i(\succ)$. Hence, it holds that $\sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) \leq R(\succ) \cdot \binom{m}{2}$ for all rankings $\succ$. Condition (3) of rank-priceability holds since $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) = \sum_{\succ \in \mathcal{R}} \pi_i(\succ, x_j) \leq 1$ for all $(x_i, x_j) \in A(\rhd)$. Finally, Condition (4) of rank-priceability is true because the total remaining budget is at most $\frac{1}{2}$. In more detail, the remaining budget in the $(m-1)$-the round is $\binom{m}{2} - \sum_{i=1}^{m-2}(m-i) = 1$ and we spent at least half of this budget in the last step. Thus, $\sum_{\succ \in \mathcal{R}} \sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) > \binom{m}{2} - 1$ and $\rhd$ is indeed rank-priceable for R. □

Theorem 5. *The Flow-adjusting Borda rule is pair-priceable.*

Proof. Fix some profile R and let $\rhd = x_1, \dots, x_m$ denote the ranking selected by FB. Further, let $b_i(\succ)$ denote the budgets of the ranking $\succ$ in the i-th round and let $X_i = \{x_i, \dots, x_m\}$. Furthermore, let G_{x_i} denote the flow network of FB in the i-th round and f_i the maximum flow chosen for G_i. We will show that the payment scheme π given by $\pi(\succ, (x_i, x_j)) = f_i(v_\succ, v_{x_j})$ for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in A(\rhd)$ satisfies all conditions of pair-priceability.

Condition (1): We first note that $\pi(\succ, (x_i, x_j)) = 0$ if $x_j \succ x_i$ because there is no edge from $\succ$ to x_j in the flow network G_{x_i} in this case. Furthermore, if $x_i \succ x_j$, it holds that $f_i(\succ, x_j) \leq 1$ because the capacity from v_{x_j} to the source is 1. Hence, it follows for all rankings $\succ$ and pairs of candidates (x_i, x_j) that $\pi(\succ, (x_i, x_j)) \leq u(\succ, x_i, \{x_i, x_j\})$.

Condition (2): This condition is true because $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ for all $\succ \in \mathcal{R}$. Moreover, we note that $b_i(\succ) \geq 0$ for all rankings $\succ$ and rounds i because it is never possible to decrease the budget of a ranking by more than $b_i(\succ)$. This is encoded in our flow network as the capacity from the sink to a ranking vertex $v_\succ$ is $b_i(\succ)$. Consequently, we have that $\sum_{(x_i, x_j) \in A(\rhd)} \pi(\succ, (x_i, x_j)) \leq R(\succ) \cdot \binom{m}{2}$ for all $\succ \in \mathcal{R}$.

Condition (3): It is immediate from the construction of the flow network G_{x_i} that $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) \leq 1$ for all $(x_i, x_j) \in A$ because the capacity of the edge from v_{x_j} to the sink t has capacity 1. Hence, it holds for the outflow of x_j that $f_i(v_{x_j}, t) \leq 1$ which, in turn, implies that $\sum_{\succ \in \mathcal{R}} f_i(v_\succ, v_{x_j}) \leq 1$. Since $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) = \sum_{\succ \in \mathcal{R}} f_i(\succ, x_j)$, this proves the third condition of pair-priceability.

Condition (4): For this condition, we need to show that the remaining budget after the execution of the Flow-adjusting Borda rule is less than 1. To this end, we will first show that for all rounds $i \in \{1, \dots, m-3\}$, the maximum flow in G_{x_i} has value $m-i$. To show this, fix such a round i and suppose that our claim is true for all rounds $j < i$. If $i = 1$, this assumption is true as there were no previous rounds and it will hold inductively for $i > 1$. We first note that the total budget in this round is

$$\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-1)m}{2} - \sum_{j=1}^{i-1} m - j = \sum_{j=1}^{m-i} j = \frac{(m-i)(m-i+1)}{2}.$$

Next, assume for contradiction that the maximum flow in $G_{x_i} = (V, E, c)$ has a value not equal to $m - i$. We note that no flow in G_{x_i} can have a value of more than $m - i$ because the capacities of all edges pointing to the sink t is $\sum_{y \in X_i \setminus \{x_i\}} c(v_y, t) = m - i$. Hence, our assumption means that the maximum flow has a value strictly less than $m - i$. By the maximum flow-MinCut equivalence, this means that there is an (s, t)-cut $S = (T, V \setminus T)$ in G_{x_i} such that $\sum_{(v,w) \in E \colon v \in T, w \in V \setminus T} c(v, w) < m - i$. It follows from this insight that S does not cut any edge connecting a ranking vertex and a candidate vertex because all of these edges have unbounded capacity. Furthermore, let $Z = \{y \in X_i \setminus \{x_i\} \colon (v_y, t) \in T \vee (v_y, t) \in V \setminus T\}$ denote the set of candidates such that the edge from the corresponding vertex to the sink is not separated by S. We note that $Z \neq \emptyset$ because otherwise, S cuts all edges from candidate vertices to the source and thus has a weight of $m - i$. Next, let $\bar{\mathcal{R}}$ denote the set of rankings $\succ$ such that there is an edge from $v_\succ$ to a candidate vertex v_y with $y \in Z$. All edges from the source to the ranking vertices $v_\succ$ with $\succ \in \bar{\mathcal{R}}$ have been cut as S does otherwise not disconnect s and t. Furthermore, the total cost for cutting these edges is less than $|Z|$ because S would have a value of at least $m - i$ otherwise. Put differently, there is a set of candidates Z such that the rankings in $\bar{\mathcal{R}} = \{\succ \in \mathcal{R} \colon \exists y \in Z \colon x_i \succ y\}$ have a total budget of less than $|Z|$.

We will show that this observation contradicts that x_i maximizes $U(b_i, x, X_i)$ when $i \geq m - 3$. To this end, we define by $\mathcal{R}^{Z \succ x_i}$ the set of rankings that prefer all candidates in a given set Z to x. Letting $z = |Z|$, our previous insights show that

$$\sum_{\succ \in \mathcal{R}^{Z \succ x}} b_i(\succ) > \frac{(m-i)(m-i+1)}{2} - z$$

because $\sum_{\succ \in \mathcal{R}^{Z \succ x}} b_i(\succ) + \sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$. Since all rankings in $\mathcal{R}^{Z \succ x}$ prefer all candidates in Z to x_i, it holds that

$$\begin{aligned}\sum_{y \in Z} \sum_{\succ \in \mathcal{R}^{Z \succ x}} b_i(\succ)\, (u(\succ, x_i, X_i) - u(\succ, y, X_i)) &\leq - \sum_{\succ \in \mathcal{R}^{Z \succ x}} b_i(\succ) \sum_{j=1}^{z} j \\ &< - \left(\frac{(m-i)(m-i+1)}{2} - z \right) \cdot \frac{z(z+1)}{2}.\end{aligned}$$

On the other hand, in the best case, it holds for all rankings $\succ \in \bar{\mathcal{R}}$ with $b_i(\succ) > 0$ that x_i is top-ranked and the candidates in Z are bottom-ranked. As the maximal Borda score with $m - i + 1$ candidates is $m - i$, we derive that

$$\begin{aligned}\sum_{y \in Z} \sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ)\, (u(\succ, x_i, X_i) - u(\succ, y, X_i)) &\leq \sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) \sum_{i=1}^{z} (m - i + 1 - z) \\ &< z \cdot \left((m-i+1)z - \frac{z(z+1)}{2} \right).\end{aligned}$$

Finally, since x_i maximizes $\sum_{\succ\in\mathcal{R}} b_i(\succ)u(\succ,x_i,X_i)$, it holds that $\sum_{\succ\in\mathcal{R}} b_i(\succ)u_i(\succ,x,X_i) \geq \sum_{\succ\in\mathcal{R}} b_i(\succ)u(\succ,z,X-i)$ for all $z\in Z$. By summing up over the candidates in Z, we hence get that

$$
\begin{aligned}
&\sum_{y\in Z}\sum_{\succ\in\mathcal{R}} b_i(\succ)(u(\succ,x_i,X_i)-u(\succ,y,X))\\
&=\sum_{y\in Z}\sum_{\succ\in\mathcal{R}^{Z\succ x}} b_i(\succ)(u(\succ,x_i,X_i)-u(\succ,y,X_i))+\sum_{y\in Z}\sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)(u(\succ,x_i,X_i)-u(\succ,y,X_i))\\
&<-\left(\frac{(m-i)(m-i+1)}{2}-z\right)\cdot\frac{z(z+1)}{2}+z\left((m-i+1)z-\frac{z(z+1)}{2}\right)\\
&=(m-i+1)z^2-\frac{(m-i)(m-i+1)z(z+1)}{4}\\
&=(m-i+1)z\left(z-\frac{(m-i)(z+1)}{4}\right)
\end{aligned}
$$

Now, if $i\leq m-4$ and thus $m-i\geq 4$, it is clear that $z-\frac{(m-i)(z+1)}{4}<0$, thus showing that x_i cannot be the Borda winner. Next, if $i=m-3$, we have that $z-\frac{(m-i)(z+1)}{4}=z-\frac{3(z+1)}{4}=\frac{z}{4}-\frac{3}{4}\leq 0$ because $z\leq 3$ if only four candidates are remaining. Combined with our previous inequality, we get a contradiction to the fact that x_i maximizes the Borda score. This shows that our initial assumption is wrong and there is indeed a maximum flow of value $m-i$ when $i\in\{1,\dots,m-3\}$.

By our analysis so far, we conclude that the total budget $\sum_{\succ\in\mathcal{R}} b_i(\succ)=3$ if $i=m-2$. Furthermore, we note that we are left with three candidates $X_{m-2}=\{x_{m-2},x_{m-1},x_m\}$ when $i=m-2$. We will next prove that, in this round, the budget decreases by at least 1.5. Assume for contradiction that this is not the case and consider again a maximum flow f_{m-2} in our flow network $G_{x_{m-2}}$. Our assumption means that the value of f_{m-2} is less than 1.5. Now, first suppose that $f_{m-2}(v_{x_{m-1}},t))<1$ and $f_{m-2}(v_{x_m},t))<1$, i.e., the outflow of both candidate vertices is less than the capacity of these edges. Since f_{m-2} is a maximum flow, this means that the capacity for all edges from the sink to the ranking vertices $v_\succ$ for rankings such that $x_{m-2}\succ x_{m-1}$ or $x_{m-2}\succ x_m$ is exhausted. Moreover, since the value of f_{m-2} is less than 1.5, this means that the total budget of these rankings is less than 1.5. Next, it holds that $u(x_{m-2},\succ,X_{m-2})\leq 2$ for all rankings $\succ$, so we derive that $U(b_{m-2},x_{m-2},X_{m-2})<3$. However, it is simple to compute that $\sum_{x\in X_{m-2}}\sum_{\succ\in\mathcal{R}} b_{m-2}(\succ)u(\succ,x,X_{m-2})=9$, so there is a candidate with a total Borda score of at least 3, thus contradicting that x_{m-2} maximizes this score.

For the second case, we assume without loss of generality that $f_{m-2}(v_{x_m},t)=1$. This implies that $f_{m-2}(v_{x_{m-1}},t)<0.5$ since the inflow of t equals the value of f_{m-2}. Now, suppose there is a ranking $\succ$ such that $b_{m-2}(\succ)>0$ and x_{m-2} is top-ranked in $\succ$. If $f_{m-2}(v_\succ,v_{x_m})>0$, we can redistribute the flow from this edge to $(v_\succ,v_{x_{m-1}})$ without reducing the value of f_{m-2}. Hence, this gives us another maximum flow and the insights of the previous paragraph yield a contradiction. Thus, we conclude that $f_{m-2}(v_\succ,v_{x_m})=0$ for all rankings $\succ$ in which x_{m-2} is top-ranked. Using again the maximality of f_{m-2}, this means that the total budget of rankings $\succ$ with $x_{m-2}\succ x_{m-1}$ is less than 0.5 as we could otherwise increase the flow through x_{m-1}. We will show that this means that x_{m-1} has a higher Borda score than x_{m-2}. For this, we observe that the rankings $\succ$ with $x_{m-2}\succ x_{m-1}$ give at most two points more to x_{m-2} than to x_{m-1}, whereas every other ranking gives at least one more point to x_{m-1} than to x_{m-2}. Thus, we conclude that

$$\begin{aligned}&U(b_{m-2}, x_{m-2}, X_{m-2}) - U(b_{m-2}, x_{m-1}, X_{m-2})\\ &\leq 2 \sum_{\succ \in \mathcal{R}\colon x_{m-2} \succ x_{m-1}} b_{m-2}(\succ) - \sum_{\succ \in \mathcal{R}\colon x_{m-1} \succ x_{m-2}} b_{m-2}(\succ)\\ &< 2 \cdot 0.5 - 2.5\\ &< 0.\end{aligned}$$

This is again a contradiction to the fact that x_{m-2} maximizes $U(b_i, x, X_{m-2})$, so we get that we indeed reduce the total budget by at least 1.5. Hence, the remaining budget for the last round is at most $3 - 1.5 = 1.5$

Finally, in the last round, the candidate maximizing $U(b_{m-1}, x, X_{m-1})$ is simply the winner of the majority vote between x_{m-1} and x_m. It is thus straightforward that we reduce the total budget by at least half, so we end up with a budget of at most $\frac{3}{4}$. This proves the last condition of pair-priceability. □

Theorem 6. *Let R be a profile on m candidates and $\rhd = \mathsf{FB}(R)$ the ranking chosen by the Flow-adjusting Borda rule. It holds for every subprofile S of R that*

$$\frac{1}{|S|} \sum_{\succ \in \mathcal{R}} S(\succ) u(\succ, \rhd) \geq \binom{m}{2} \cdot \frac{|S|}{4} - \frac{3}{16}.$$

Proof. Fix a profile R and let $\rhd = x_1, \dots, x_m$ denote the ranking chosen by FB. We will closely follow the proof of Theorem 2 and thus define by $b_i(\succ)$ the budget of ranking $\succ$ in the i-th round of the Flow-adjusting Borda rule. In particular, $b_1(\succ) = R(\succ) \cdot \binom{m}{2}$ for all rankings $\succ \in \mathcal{R}$. Furthermore, let f_i denote the maximum flow chosen in the i-th round of FB and define the cost per utility ratio ρ_i by $\max_{\succ \in \mathcal{R}\colon b_i(\succ) > 0} \frac{f_i(s, v_\succ)}{b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})}$. Lastly, we will throughout the proof assume that $\frac{0}{0} = 0$ to avoid trivial corner cases.

We will show this theorem in two steps. First, we will show that $\rho_i \leq \frac{4}{(m-i)(m-i+1)}$ for all $i \in \{1, \dots, m-3\}$, $\rho_{m-2} \leq 1$, and $\rho_{m-1} \leq 1$. Based on this insight, we will prove the theorem in a second step.

Step 1: We start by showing our upper bounds on ρ_i. To this end, we fix a round i and let $G_{x_i} = (V, E, c)$ denote the flow network used by FB in this round. It holds for every ranking $\succ$ that $f_i(s, v_\succ) \leq b_i(\succ)$ since $c(s, v_\succ) = b_i(\succ)$. Hence, we derive that $\frac{f_i(s, v_\succ)}{b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})} \leq \frac{1}{u(\succ, x_i, \{x_i, \dots, x_m\})} \leq 1$ if $u(\succ, x_i, \{x_i, \dots, x_m\}) > 0$. On the other, if $u(\succ, x_i, \{x_i, \dots, x_m\}) = 0$, then $\succ$ bottom-ranks x_i among $\{x_i, \dots, x_m\}$. Thus, $\succ$ has no outgoing edge in G_{x_i} and $f(s, v_\succ) = 0$. This implies that $\frac{f_i(s, v_\succ)}{b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})} = 0$ if $u_i(\succ, x_i, \{x_i, \dots, x_m\}) = 0$ (or, put differently, we can ignore $\succ$ as it does not pay anything in this case). It follows that $\rho_i \leq 1$ for all rounds i, thus proving our upper bounds for ρ_{m-2} and ρ_{m-1}.

Next, we suppose that $i \in \{1, \dots, m-3\}$ and aim to show that $\frac{f_i(s, v_\succ)}{b_i(\succ) \cdot u_i(\succ)} \leq \frac{4}{(m-i)(m-i+1)}$. To this end, we will prove that the flow network G_{x_i} admits a flow f_i^* with value $m - i$ such that $f_i^*(s, v_\succ) \leq \frac{4 b_i(\succ) u(\succ, x_i, \{x_i, \dots, x_m\})}{(m-i)(m-i+1)}$ for all $\succ \in \mathcal{R}$. For this, consider the modified flow network $G'_{x'_i} = (V, E, c')$, which uses the same vertices and edges as G_{x_i} but has different capacities. Specifically, we set $c'(s, v_\succ) = b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})$ for all ranking vertices, $c'(v_\succ, v_x)$ is still unbounded for all rankings $\succ$ and candidates x with $x_i \succ x$, and $c'(v_x, t) = \frac{(m-i)(m-i+1)}{4}$ for all candidates $x \in \{x_{i+1}, \dots, x_m\}$. We will show that G'_{x_i} permits a flow f'_i of value $\frac{(m-i)^2(m-i+1)}{4}$. Based on f'_i, we will then define the flow f_i^* by $f_i^*(e) = \frac{4 f'_i(e)}{m(m-1)}$ for all $e \in E$. We claim that f_i^* is feasible for G_{x_i} and satisfies that $\frac{f_i^*(s,v_\succ)}{b_i(\succ)\cdot u(\succ,x_i,\{x_i,\dots,x_m\})} \leq \frac{4}{(m-i)(m-i+1)}$ for all $\succ \in \mathcal{R}$. For the feasibility, we observe for every edge $e \in E$ that $f_i^*(e) = \frac{4f_i'(e)}{(m-i)(m-i+1)} \leq \frac{4c'(e)}{(m-i)(m-i+1)} \leq c(e)$. In more detail, for the edges from candidates vertices v_x to the sink t, this holds as $c'(v_x,t) = \frac{(m-i)(m-i+1)}{4}$ and $c(v_x,t) = 1$. For edges from the source s to ranking vertices $v_\succ$, our claim holds as $\frac{4c'(s,v_\succ)}{(m-i)(m-i+1)} = \frac{4b_i(\succ)u(\succ,x_i,\{x_i,\dots,x_m\}}{(m-i)(m-i+1)} \leq b_i(\succ) = c(s,v_\succ)$. For the second step, we use that $u(\succ,x_i,\{x_i,\dots,x_m\}) \leq m-i$ and that $4 \leq m-i+1$ as $i \leq m-3$. Furthermore, it holds by definition that $f_i^*(x,v_\succ) \leq \frac{4b_i(\succ)u(\succ,x_i,\{x_i,\dots,x_m\})}{(m-i)(m-i+1)}$ for all ranking $\succ$, so $\frac{f_i^*(s,v_\succ)}{b_i(\succ)\cdot u_i(\succ,x_i,\{x_i,\dots,x_m\})} \leq \frac{4}{(m-i)(m-i+1)}$ for all $\succ \in \mathcal{R}$.

Now, we assume for contradiction that our modified flow network G'_{x_i} does not permit a flow of value $\frac{(m-i)^2(m-i+1)}{4}$. By the maximum flow-MinCut equivalence, this means that there is an (s,t)-cut $(T, V \setminus T)$ in G'_{x_i} whose total weight is less than $\frac{(m-i+1)(m-i)^2}{4}$. Let Z denote the set of candidates for which the candidate vertex is not separated from the sink by $(T, V \setminus T)$, i.e., Z is the set of candidates x such that $v_x \in T$ if and only if $t \in T$. Since there are $(m-i)$ candidate vertices and all edges (v_x,t) have a weight of $\frac{(m-i)(m-i+1)}{4}$, we derive that $Z \neq \emptyset$ as $(T, V \setminus T)$ would have a value of at least $\frac{(m-i+1)(m-i)^2}{4}$ otherwise. Moreover, $(T, V \setminus T)$ cannot disconnect any edge from a ranking node to a candidate node as these have unbounded capacity. Finally, let $\bar{\mathcal{R}}$ denote the set of rankings $\succ$ such that $(v_\succ, v_x) \in E$ for a candidate $x \in Z$. All edges from the sink to the ranking vertices $v_\succ$ for $\succ \in \bar{\mathcal{R}}$ must be disconnected as there is otherwise still a path from s to t in G'_{x_i}. Moreover, the total capacities of these edges is less than $|Z|\frac{(m-i)(m-i+1)}{4}$. Otherwise, the weight of $(T, V \setminus T)$ is at least $\frac{(m-i+1)(m-i)^2}{4}$ because we cut $(m-i-|Z|)$ edges from candidate vertices to the sink, each of which has a weight of $\frac{(m-i+1)(m-i)}{4}$.

In summary, this analysis shows that there is a set of candidates $Z \subseteq \{x_{i+1},\dots,x_m\}$ such that $\sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)u(\succ,x_i,\{x_i,\dots,x_m\}) < z\frac{(m-i+1)(m-i)}{4}$, where $z = |Z|$ and $\bar{\mathcal{R}} = \{\succ \in \mathcal{R} \colon \exists y \in Z \colon x_i \succ y\}$. We will show that this contradicts with the fact that x_i maximizes the $U(b_i, x_i, \{x_i,\dots,x_m\}$. To this end, let $\varphi = \sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)$ and first assume that $\varphi \geq \frac{(m-i+1)(m-i)}{4}$. Because each ranking $\succ \in \mathcal{R}\setminus\bar{\mathcal{R}}$ prefers all candidates in Z to x_i, x_i receives a score of at most $m-z-i$ from each of these rankings. Since we have shown in the proof of Theorem 5 that the total remaining budget is $\sum_{\succ\in\mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$, this means that $\sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)u(\succ,x_i,\{x_i,\dots,x_m\}) \leq (\frac{(m-i)(m-i+1)}{2} - \varphi)(m-z-i)$. Combined with the assumption that $\sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)u(\succ,x_i,\{x_i,\dots,x_m\}) < z\frac{(m-i)(m-i+1)}{4}$, we now derive that

$$
\begin{aligned}
&\sum_{\succ\in\mathcal{R}} b_i(\succ)u_i(\succ,x_i,\{x_i,\dots,x_m\}) \\
&= \sum_{\succ\in\mathcal{R}\setminus\bar{\mathcal{R}}} b_i(\succ)u(\succ,x,C) + \sum_{\succ\in\bar{\mathcal{R}}} b_i(\succ)u(\succ,x,C) \\
&< \left(\frac{(m-i)(m-i+1)}{2} - \varphi\right)(m-z-i) + z\frac{(m-i)(m-i+1)}{4} \\
&\leq \frac{(m-i)^2(m-i+1)}{4}.
\end{aligned}
$$

In the last step, we use that $\varphi \geq \frac{(m-i)(m-i+1)}{4}$ and thus $\frac{(m-i)(m-i+1)}{2} - \varphi \leq \frac{(m-i)(m-i+1)}{4}$. We next note that the average Borda score with respect to b_i is

$$\begin{aligned}\frac{1}{m-i+1} \sum_{y \in \{x_i, \dots, x_m\}} \sum_{\succ \in \mathcal{R}} b_i(\succ) u(\succ, y, \{x_i, \dots, x_m\}) &= \frac{1}{m-i+1} \sum_{\succ \in \mathcal{R}} b_i(\succ) \sum_{j=0}^{m-i} j \\ &= \frac{(m-i)^2(m-i+1)}{4}.\end{aligned}$$

However, this means that there is a candidate y with $\sum_{\succ \in \mathcal{R}} b_i(\succ) u(\succ, y, \{x_i, \dots, x_m\}) \geq \frac{(m-i)^2(m-i+1)}{4} > U(\succ, x_i, \{x_i, \dots, x_m\})$, which contradicts that x_i is selected by FB in this step.

As the second case, suppose that $\varphi < \frac{m(m-1)}{4}$. Since x_i maximizes the total Borda score with respect to x_i, it holds for every candidate $y \in Z$ that

$$\begin{aligned}0 \leq &\sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ)(u(\succ, x_i, \{x_i, \dots, x_m\}) - u(\succ, y, \{x_i, \dots, x_m\})) \\ &+ \sum_{\succ \in \mathcal{R} \setminus \bar{\mathcal{R}}} b_i(\succ)(u(\succ, x_i, \{x_i, \dots, x_m\}) - u(\succ, y, \{x_i, \dots, x_m\})).\end{aligned}$$

This implies further that

$$\begin{aligned}0 \leq &\frac{1}{z} \sum_{\succ \in \bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ)(u(\succ, x_i, \{x_i, \dots, x_m\}) - u(\succ, y, \{x_i, \dots, x_m\})) \\ &+ \frac{1}{z} \sum_{\succ \in \mathcal{R} \setminus \bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ)(u(\succ, x_i, \{x_i, \dots, x_m\}) - u(\succ, y, \{x_i, \dots, x_m\})).\end{aligned}$$

Equivalently, this means that

$$\begin{aligned}\sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) u(\succ, x_i, \{x_i, \dots, x_m\}) \geq &\frac{1}{z} \sum_{\succ \in \bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ) u(\succ, y, \{x_i, \dots, x_m\}) \\ &+ \frac{1}{z} \sum_{\succ \in \mathcal{R} \setminus \bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ)(u(\succ, y, \{x_i, \dots, x_m\}) - u(\succ, x_i, \{x_i, \dots, x_m\})).\end{aligned}$$

Next, since $\sum_{y \in Z} u(\succ, y, \{x_i, \dots, x_m\}) \geq \sum_{j=0}^{z-1} j$ for every ranking $\succ$, it follows that

$$\frac{1}{z} \sum_{\succ \in \bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ) u(\succ, y, \{x_i, \dots, x_m\}) \geq \frac{1}{z} \sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) \sum_{j=0}^{z-1} j = \varphi \frac{z-1}{2}.$$

Furthermore, every ranking $\succ \in \mathcal{R} \setminus \bar{\mathcal{R}}$ ranks all candidates in Z ahead of x_i, so $\sum_{y \in Z} u(\succ, y, \{x_i, \dots, x_m\}) - u(\succ, x_i, \{x_i, \dots, x_m\}) \geq \sum_{j=1}^{z} j$ for each $\succ \in \mathcal{R} \setminus \bar{\mathcal{R}}$. Based on this insight and the fact that $\sum_{\succ \in \mathcal{R}} b_i(\succ) = \frac{(m-i)(m-i+1)}{2}$, we compute that

$$\begin{aligned}
&\frac{1}{z} \sum_{\succ \in \mathcal{R}\setminus\bar{\mathcal{R}}} \sum_{y \in Z} b_i(\succ)(u(\succ, y, \{x_i, \dots, x_m\}) - u(\succ, x_i, \{x_i, \dots, x_m\})) \\
&\geq \frac{1}{z} \sum_{\succ \in \mathcal{R}\setminus\bar{\mathcal{R}}} b_i(\succ) \sum_{j=1}^{z} j \\
&= \frac{1}{z} \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) \left(\sum_{j=1}^{z} (m-i+1-j) - z(m-i-z) \right) \\
&= \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) \cdot \left((m-i+1) - \frac{z+1}{2} \right) - \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) (m-i-z)
\end{aligned}$$

Putting our inequalities together, this means that

$$\begin{aligned}
&\sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) u(\succ, x_i, \{x_i, \dots, x_m\}) \\
&\geq \varphi \frac{z-1}{2} + \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) \cdot \left((m-i+1) - \frac{z+1}{2} \right) \\
&\quad - \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) (m-i-z) \\
&= \frac{(m-i)(m-i+1)}{2} \cdot \frac{z-1}{2} + \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) \cdot (m-i+1-z) \\
&\quad - \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) (m-i-z) \\
&= \frac{(m-i)(m-i+1)}{2} \cdot \frac{z-1}{2} + \left(\frac{(m-i)(m-i+1)}{2} - \varphi \right) \\
&\geq \frac{m(m-1)}{4} z.
\end{aligned}$$

Here, the last inequality follows by using that $\varphi \leq \frac{m(m-1)}{4}$. This directly disproves the assumption that $\sum_{\succ \in \bar{\mathcal{R}}} b_i(\succ) u(\succ, x_i, \{x_i, \dots, x_m\}) < \frac{m(m-1)}{4} z$, so we also showed our claim in this case.

Step 2: We are now ready to prove the theorem and thus fix an arbitrary subprofile S of R. We will closely follow the proof of Theorem 2 and thus define $b_i^S(\succ) = \frac{S(\succ)}{R(\succ)} b_i(\succ) = S(\succ) \cdot \binom{m}{2}$ for all $i \in \{1, \dots, m\}$ and $\succ \in \mathcal{R}$. Next, we let $c_i^S(\succ) = b_i^S(\succ) - b_{i+1}^S(\succ)$ denote the payment made by the ranking $\succ$ with respect to its budget in S and by $C_i^S = \sum_{\succ \in \mathcal{R}} c_i^S(\succ)$ the total payment made by the subprofile S in the i-th round.

Now, fix a round $i \in \{1, \dots, m-1\}$ in the execution of the Flow-adjusting Borda rule. We first observe that for every ranking $\succ$, it holds that $b_i(\succ) - b_{i+1}(\succ) = f_i(\succ)$ as the computed flow determines the payment. Furthermore, by the definition of ρ_i, we have that $\frac{f_i(s, v_\succ)}{b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})} \leq \rho_i$. Equivalently, this means that $\frac{b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\})}{f_i(s, v_\succ)} \geq \frac{1}{\rho_i}$. By combining our insight, it follows that $b_i(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\}) \geq \frac{b_i(\succ) - b_{i+1}(\succ))}{\rho_i}$. Furthermore, by multiplying both sides with $\frac{S(\succ)}{R(\succ)}$, we get that $b_i^S(\succ) \cdot u(\succ, x_i, \{x_i, \dots, x_m\}) \geq \frac{c_i^S(\succ)}{\rho_i}$. Finally, summing over all the rankings

$\succ \in \mathcal{R}$, we conclude that

$$\frac{C_i^S}{\rho_i} \leq \sum_{\succ \in \mathcal{R}} b_i^S(\succ) u(\succ, x_i, \{x_1, \dots, x_m\}) \leq \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, x_i, \{x_1, \dots, x_m\}).$$

Moreover, by summing over all rounds $i \in \{1, \dots, m-1\}$, we infer that

$$\sum_{i=1}^{m-1} \frac{C_i^S}{\rho_i} \leq \sum_{i=1}^{m-1} \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, x_i, \{x_i, \dots, x_m\}) = \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, \rhd).$$

In turn, our upper bounds on ρ_i proven in Step 1 show that

$$\sum_{i=1}^{m-3} \frac{C_i^S (m-i)(m-i+1)}{4} + C_{m-2}^S + C_{m-1}^S \leq \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, \rhd).$$

Next, let $C = \sum_{i=1}^{m-1} C_i^S$ denote the total payments made by S and note that $C \geq \sum_{\succ \in \mathcal{R}} b_1^S(\succ) - \frac{3}{4}$ because we have shown in the proof of Theorem 5 that the total remaining budget of FB is at most $\frac{3}{4}$. Moreover, let k denote the largest integer such that $C \geq \frac{k(k+1)}{2}$. Now, since the coefficients of our sum are weakly decreasing as i increases, we minimize this term by assuming that C is only distributed at the late payments. However, in each round i, it is only possible to pay at most $m-i$. Hence, the left-hand sum of the above inequality is minimized if $C_{m-i} = m-i$ for all $i \in \{m-1, \dots, m-k\}$ and $C_{m-k-1} = C - \frac{k(k+1)}{2}$.

We next proceed with a case distinction regarding C (resp. k) and first suppose that $C \leq 3$. In this case, we have in the worst-case that $C_i^S = 0$ for all $i \in \{1, \dots, m-3\}$, so we can simplify our inequality to $C \leq \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, \rhd)$. Further, by using that $C \geq |S| \cdot \binom{m}{2} - \frac{3}{4}$ and dividing by $|S| \cdot \binom{m}{2}$, this means that

$$1 - \frac{3}{2m(m-1) \cdot |S|} \leq \frac{1}{|S|} \cdot \binom{m}{2}^{-1} \cdot \sum_{\succ \in \mathcal{R}} b_1^S(\succ) u(\succ, \rhd) = \frac{1}{|S|} \sum_{\succ \in \mathcal{R}} S(\succ) u(\succ, \rhd).$$

Now, we first note that the bound of the theorem is trivial if $|S| \leq \frac{3}{4} \cdot \binom{m}{2}^{-1}$ because then $\binom{m}{2} \cdot \frac{|S|}{4} - \frac{3}{16} \leq 0$. We hence assume that $|S| > \frac{3}{4}$. Moreover, our assumption that $C \leq 3$ implies that $|S| \leq 3 + \frac{3}{4}$. We will now show that for all these values of $|S|$ that $\binom{m}{2} \cdot \frac{|S|}{4} - \frac{3}{16} \leq 1 - \frac{3}{2m(m-1) \cdot |S|}$. By subtracting $1 - \frac{3}{2m(m-1) \cdot |S|}$ form both sides and multiplying with $16|S| \cdot \binom{m}{2}$, we infer that this is equivalent to

$$4\binom{m}{2}^2 |S|^2 - 19\binom{m}{2}|S| + 12 \leq 0.$$

It can now be checked that $4\binom{m}{2}^2 |S|^2 - 19\binom{m}{2}|S| + 12 = 0$ if $|S| = \frac{3}{4}\binom{m}{2}^{-1}$ or $|S| = 4\binom{m}{2}^{-1}$. As a quadratic function grows from its minimum, this proves our inequality for $\frac{3}{4}\binom{m}{2}^{-1} \leq |S| \leq (3 + \frac{3}{4})\binom{m}{2}^{-1}$, as required.

As the second case, suppose that $C > 3$ and thus $k \geq 2$. In this case, we have that

$$1 + 2 + \sum_{i=m-k}^{m-3} \frac{(m-k)(m-k)(m-k+1)}{4} + \frac{(C - \frac{k(k+1)}{2})(m - (m-k-1))(m - (m-k-1) + 1)}{4}$$
$$\leq C_{m-1}^S + C_{m-2}^S + \sum_{i=1}^{m-3} \frac{C_i^S (m-i)(m-i+1)}{4}.$$

Moreover, since $\frac{2\cdot 2\cdot 3}{4} + \frac{1\cdot 1\cdot 2}{4} = \frac{7}{2}$, we can rewrite the left side of this inequality by

$$\sum_{i=m-k}^{m-1} \frac{(m-k)(m-k)(m-k+1)}{4} + \frac{(C - \frac{k(k+1)}{2})(m-(m-k-1))(m-(m-k-1)+1)}{4} - \frac{1}{2}$$
$$= \sum_{i=1}^{k} \frac{i^2(i+1)}{4} + \frac{(C - \frac{k(k+1)}{2})(k+1)(k+2)}{4} - \frac{1}{2}$$

Next, let $\ell = C - \frac{k(k+1)}{2}$. As noted in the proof of Theorem 2, it holds that

$$\begin{aligned}
&\sum_{i=1}^{k} i^2(i+1) + \ell(k+1)(k+2) \\
&= \frac{k^4}{4} + \frac{5k^3}{6} + \frac{3k^2}{4} + \frac{k}{6} + \ell(k+1)(k+2) \\
&= \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell(k+1)\right) + \left(\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + \ell(k+1)\right)
\end{aligned}$$

Now, we first note that $\ell \leq k+1$, so $\ell(k+1) \geq \ell^2$. Further, as $k \geq 2$, it holds that $\frac{k^3}{3} + \frac{k}{6} \geq \frac{k^2}{2} + 1$ and $k^2 \geq \frac{k(k+1)}{2}$. Hence, we derive that

$$\begin{aligned}
\sum_{i=1}^{k} i^2(i+1) + \ell(k+1)(k+2) &\geq \left(\frac{k^4}{4} + \frac{2k^3}{4} + \frac{k^2}{4} + \ell k(k+1)) + \ell^2\right) + \left(k^2 + 1 + \ell(k+1)\right) \\
&\geq \left(\frac{k(k+1)}{2} + \ell\right)^2 + \left(\frac{k(k+1)}{2} + \ell\right) + 2 \\
&= C(C+1) + 2.
\end{aligned}$$

Substituting this into our original inequality shows that $\frac{C(C+1)}{4} \leq \sum_{\succ \in \mathcal{R}} b_i^S(\succ) u(\succ, \rhd)$. Finally, from here on, we can complete the proof analogously to the proof of Theorem 2. □